

On the General Ericksen–Leslie System: Parodi’s Relation, Well-posedness and Stability

HAO WU ^{*}, XIANG XU [†] and CHUN LIU [‡]

January 20, 2013

Abstract

In this paper we investigate the role of Parodi’s relation in the well-posedness and stability of the general Ericksen–Leslie system modeling nematic liquid crystal flows. First, we give a formal physical derivation of the Ericksen–Leslie system through an appropriate energy variational approach under Parodi’s relation, in which we can distinguish the conservative/dissipative parts of the induced elastic stress. Next, we prove global well-posedness and long-time behavior of the Ericksen–Leslie system under the assumption that the viscosity μ_4 is sufficiently large. Finally, under Parodi’s relation, we show the global well-posedness and Lyapunov stability for the Ericksen–Leslie system near local energy minimizers. The connection between Parodi’s relation and linear stability of the Ericksen–Leslie system is also discussed.

Keywords: Liquid crystal flows, Ericksen–Leslie System, Parodi’s relation, uniqueness of asymptotic limit, stability.

AMS Subject Classification: 35B40, 35B41, 35Q35, 76D05.

1 Introduction

Liquid crystal is often viewed as the fourth state of the matter besides the gas, liquid and solid, or as an intermediate state between liquid and solid. It possesses none or partial positional order but displays an orientational order at the same time. The nematic phase is the simplest among all liquid crystal phases and is close to the liquid phase. The molecules float around as in a liquid phase, but have the tendency of aligning along a preferred direction due to their orientation. The hydrodynamic theory of liquid crystals due to Ericksen and Leslie was developed around 1960’s [12, 13, 25, 26]. Earlier attempts on rigorous mathematical analysis of the Ericksen–Leslie system were made recently [32] (see [28, 30, 31] for a simplified system which carried important mathematical difficulties of the original Ericksen–Leslie system, except the kinematic transport of the director field).

The full Ericksen–Leslie system consists of the following equations (cf. [14, 26, 27, 32]):

$$\rho_t + v \cdot \nabla \rho = 0, \quad (1.1)$$

$$\rho \dot{v} = \rho F + \nabla \cdot \hat{\sigma}, \quad (1.2)$$

$$\rho_1 \dot{\omega} = \rho_1 G + \hat{g} + \nabla \cdot \pi. \quad (1.3)$$

^{*}School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, 200433 Shanghai, China, Email: *haowufd@yahoo.com*.

[†]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, Email: *xuxiang@math.cmu.edu*.

[‡]Department of Mathematics, Penn State University, State College, PA 16802, Email: *liu@math.psu.edu*.

Equations (1.1)–(1.3) represent the conservation of mass, linear momentum, and angular momentum, respectively, with the anisotropic feature of liquid crystal materials exhibited in (1.3) and its nonlinear coupling in (1.2) (cf. [26, 32]). In this paper, we consider the flow of an incompressible material, namely, $\nabla \cdot v = 0$. Here, ρ is the fluid density, ρ_1 is a (positive) inertial constant, $v = (v_1, v_2, v_3)^T$ is the flow velocity, $d = (d_1, d_2, d_3)^T$ is the orientational order parameter representing the macroscopic average of the molecular directors, \hat{g} is the intrinsic force associated with d , π is the director stress, F and G are external body force and external director body force, respectively. The superposed dot denotes the material derivative $\partial_t + v \cdot \nabla$. The notations

$$\begin{aligned} A &= \frac{1}{2}(\nabla v + \nabla^T v), & \Omega &= \frac{1}{2}(\nabla v - \nabla^T v), \\ \omega &= \dot{d} = d_t + (v \cdot \nabla)d, & N &= \omega - \Omega d, \end{aligned}$$

represent the rate of strain tensor, skew-symmetric part of the strain rate, the material derivative of d (transport of center of mass) and rigid rotation part of director changing rate by fluid vorticity, respectively.

We have the following constitutive relations in the system (1.1)–(1.3) for $\hat{\sigma}$, π and \hat{g} :

$$\hat{\sigma}_{ij} = -P\delta_{ij} - \rho \frac{\partial W}{\partial d_{k,i}} d_{k,j} + \sigma_{ij}, \quad (1.4)$$

$$\pi_{ij} = \beta_i d_j + \rho \frac{\partial W}{\partial d_{j,i}}, \quad (1.5)$$

$$\hat{g}_i = \gamma d_i - \beta_j d_{i,j} - \rho \frac{\partial W}{\partial d_i} + g_i. \quad (1.6)$$

Here P is a scalar function representing the pressure. The vector $\beta = (\beta_1, \beta_2, \beta_3)^T$ and the scalar function γ (sometimes called director tension) are Lagrangian multipliers for the constraint on the length of director such that $|d| = 1$, with the Oseen–Frank energy functional W for the equilibrium configuration of a unit director field:

$$\begin{aligned} W &= \frac{k_1}{2}(\nabla \cdot d)^2 + \frac{k_2}{2}|d \times (\nabla \times d)|^2 + \frac{k_3}{2}|d \cdot (\nabla \times d)|^2 \\ &\quad + (k_2 + k_4)[\text{tr}(\nabla d)^2 - (\nabla \cdot d)^2]. \end{aligned} \quad (1.7)$$

We note that the forth term in (1.7)

$$\text{tr}(\nabla d)^2 - (\nabla \cdot d)^2 = \nabla \cdot [(\nabla d)d - (\nabla \cdot d)d]$$

is a null Lagrangian, which only depends on the value of the trace of d on the boundary (cf. [1]).

The kinematic transport of the director d (denoted by g) is given by:

$$g_i = \lambda_1 N_i + \lambda_2 d_j A_{ji} = \lambda_1 \left(N_i + \frac{\lambda_2}{\lambda_1} d_j A_{ji} \right), \quad (1.8)$$

which represents the effect of macroscopic flow field on the microscopic structure. The material coefficients λ_1 and λ_2 reflects the molecular shape (Jeffrey’s orbit [21]) and the slippery between the fluid and the particles (see discussions in Section 3). The first term of (1.8) represents the rigid rotation of the molecule, while the second term stands for the stretching of the molecule by the flow.

The stress tensor σ has the following form:

$$\sigma_{ij} = \mu_1 d_k A_{kp} d_p d_i d_j + \mu_2 N_i d_j + \mu_3 d_i N_j + \mu_4 A_{ij}$$

$$+\mu_5 A_{ik} d_k d_j + \mu_6 d_i A_{jk} d_k. \quad (1.9)$$

These (independent) coefficients μ_1, \dots, μ_6 , which may depend on material and temperature, are usually called Leslie coefficients. These coefficients are related to certain local correlations in the fluid (cf. [11]). For convenience, μ'_i 's are called viscous coefficients in later sections.

In order to handle the higher-order nonlinearities due to the nonlinear constraint $|d| = 1$ (i.e., the Lagrangian multipliers β, γ), one can introduce a penalty (or relaxation) approximation of Ginzburg–Landau type, by adding one term

$$\mathcal{F}(d) = \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2$$

in W . Physically this term can be attributed to the extensibility of the molecules. After the discussions for each $\varepsilon > 0$, we then take the limit as $\varepsilon \rightarrow 0$. This method is motivated by the work on the gradient flow of harmonic maps into the sphere (see, e.g., [5]), but whether the solution of the Ericksen–Leslie system with Ginzburg–Landau approximation converges to that of the original one with constraint $|d| = 1$ as ε tends to zero is still a challenging problem. Nevertheless, the reformulated system with penalty approximation also has natural physical interpretations. It is similar to what Leslie proposed in [27] for the flow of an anisotropic liquid with varying director length. Mathematically, it can also be related to models for nematic liquid crystals with variable degree of orientation proposed by Ericksen in [14] under specific conditions. In particular, $\{x : d(x, t) = 0\}$ represents the transition region of isotropic fluids. We refer to [30] for more discussions.

For simplicity, in this paper, we focus on the relaxation form of the elastic energy associated with d :

$$W(d) = \frac{1}{2}|\nabla d|^2 + \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2. \quad (1.10)$$

It is obvious that this choice of W corresponds to the elastically isotropic situation, i.e., taking $k_1 = k_2 = k_3 = 1, k_4 = 0$ in (1.7). The corresponding problem with general Oseen–Frank energy (1.7) can be treated in a similar way, but the argument is more involved. With the choice of the penalized energy W , we can remove the Lagrangian multipliers and set $\gamma = \beta_j = 0$. Since the inertial constant ρ_1 is usually very small (cf. [10]), we take $\rho_1 = 0$. Moreover, we assume that the density is constant (which in turn yields the incompressibility $\nabla \cdot v = 0$) and the external forces vanish, i.e., $\rho = 1, F = 0, G = 0$ (cf. [32]). Note that in the incompressible cases the assumption $F = 0$ means that there is no exterior nonconservative forces.

Thus, the full Ericksen–Leslie system (1.1)–(1.3) can be reformulated to

$$v_t + v \cdot \nabla v + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \sigma, \quad (1.11)$$

$$\nabla \cdot v = 0, \quad (1.12)$$

$$d_t + (v \cdot \nabla)d - \Omega d + \frac{\lambda_2}{\lambda_1} A d = -\frac{1}{\lambda_1} (\Delta d - f(d)), \quad (1.13)$$

where

$$f(d) = \mathcal{F}'(d) = \frac{1}{\varepsilon^2}(|d|^2 - 1)d$$

and σ is given by (1.9). We denote by $\nabla d \odot \nabla d$ the 3×3 -matrix whose (i, j) -entry is $\nabla_i d \cdot \nabla_j d$, $1 \leq i, j \leq 3$. In the following text, we just set $\varepsilon = 1$ and our results indeed hold for any arbitrary but fixed $\varepsilon > 0$. In this paper, we will focus on the bulk properties of the Ericksen–Leslie system. For this, we consider the equations (1.11)–(1.13) subject to periodic boundary conditions (i.e., in torus \mathbb{T}^3):

$$v(x + e_i, t) = v(x, t), \quad d(x + e_i, t) = d(x, t), \quad \text{for } (x, t) \in \partial Q \times \mathbb{R}^+, \quad (1.14)$$

and initial conditions

$$v|_{t=0} = v_0(x), \quad \text{with } \nabla \cdot v_0 = 0, \quad d|_{t=0} = d_0(x), \quad \text{for } x \in Q, \quad (1.15)$$

where Q is a unit square in \mathbb{R}^3 .

Due to temperature dependence of the Leslie coefficients, there exists different behavior between various coefficients (cf. [11]): μ_4 -which does not involve the alignment properties-is a rather smooth function of temperature; but all the other μ 's describe couplings between the molecule orientation and the flow, and are thus affected by a decrease in the nematic order $|d|$. In this paper, we just look at the isothermal case where μ 's are assumed to be constants. The following relations are frequently introduced in the literature (cf. [26, 27])

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \quad (1.16)$$

$$\mu_2 + \mu_3 = \mu_6 - \mu_5. \quad (1.17)$$

Relations given in (1.16) are necessary conditions in order to satisfy the equation of motion identically (cf. [26, Section 6]). (1.17) is called *Parodi's relation* (cf. [40]), which is derived from Onsager reciprocal relations expressing the equality of certain relations between flows and forces in thermodynamic systems out of equilibrium (cf. [37]). Under the assumption of Parodi's relation, we see that the dynamics of an incompressible nematic liquid crystal flow involve five independent Leslie coefficients in (1.9).

Since the mathematical structure of the Ericksen–Leslie system is quite complicated, past existing work was almost all restricted to its simplified versions (cf. [3, 30, 31, 34, 35, 44]). As far as the general Ericksen–Leslie system is concerned, there is few known result in analysis (cf. e.g., [32]). In [32], well-posedness of the general Ericksen–Leslie system (1.11)–(1.13) subject to Dirichlet boundary conditions was proved under the special assumption $\lambda_2 = 0$, which imposed an extra constraint on those Leslie coefficients. This physically indicates that the stretching due to the flow field is neglected, which is more feasible for small molecules. Mathematically this assumption brings great convenience since a weak maximum principle for $|d|$ holds (cf. [32, Theorem 3.1]). For the general system (1.11)–(1.13), the maximum principle for $|d|$ fails when $\lambda_2 \neq 0$. This leads to extra difficulties in the study of well-posedness, especially in dealing with those highly nonlinear stress terms in σ (cf. [32, 44]). Even in the 2D case, it is hard to obtain global existence of solutions without any further restriction on these viscous coefficients. This is rather different from regular Newtonian fluid cases.

Summary of results. The purpose of this paper is to study the connections between physical parameters, namely, the Leslie coefficients and the well-posedness as well as stability properties of the general Ericksen–Leslie system (1.11)–(1.13). In particular, we focus on the role of Parodi's relation (1.17). Parodi's relation is a consequence of Onsager's reciprocal relations in the microscale descriptions of liquid crystals [37, 38], which are nevertheless independent of the second law of thermodynamics. Although the physical interpretation of the reciprocal relation is related to microscopic reversibility and laws of detailed balance of microscopic dynamics [36], the thermodynamic basis of Onsager's reciprocal relations have been discussed and debated by many researchers (cf. [46]). There are evidences that for particular materials, Onsager's relations and their counterparts may serve as stability conditions (cf. [9, 46]). In this paper, we provide specific mathematical verifications for the nematic liquid crystal flow.

First, in Theorems 4.1 and 4.2, under the assumption that the fluid viscosity μ_4 is sufficiently large, we show existence and uniqueness of global solutions within suitable regularity classes and their long-time behavior (uniqueness of asymptotic limit). In this case, we see that Parodi's relation is not a necessary assumption for the well-posedness and long-time dynamics, while the large viscosity constant μ_4 plays a dominative role.

Next, without the largeness assumption on μ_4 , we first prove local wellposedness of the Ericksen–Leslie system (1.11)–(1.13) (cf. Theorem 5.1). Furthermore, in Theorem 5.2, we prove

global well-posedness and Lyapunov stability of the Ericksen–Leslie system, when the initial data is near certain equilibrium (local minimizer of the elastic energy W given by (1.10)). We see that Parodi’s relation turns out to be crucial (as a sufficient condition) in obtaining well-posedness and (nonlinear) stability of the Ericksen–Leslie system.

Finally, we demonstrate the connection between Parodi’s relation and linear stability of the original Ericksen–Leslie system (1.1)–(1.3) (namely, with the constraint $|d| = 1$). The result obtained in Theorem 6.1 indicates that without Parodi’s relation, the linearized Ericksen–Leslie system admits unstable plane wave solutions. In other words, Parodi’s relation is a necessary condition for linear stability of the Ericksen–Leslie system.

Remark 1.1. *Our results presented in this paper are stated in the three dimensional case $n = 3$. When the spatial dimension $n = 2$, if we consider the velocity field $v : \Omega \times [0, T] \rightarrow \mathbb{R}^2$ and director $d : \Omega \times [0, T] \rightarrow \mathbb{R}^2$, namely, the molecule director d is also confined in a plane, then it is easy to verify that all the results we obtained below for the 3D case also hold in 2D (sometimes even under weaker assumptions, see e.g., Remark 4.1). However, if one wishes to consider the Ericksen–Leslie system in a 2D domain $\Omega \subset \mathbb{R}^2$ but the director field d is still allowed to be a three dimensional vector, some troubles will come up. For instance, the parallel transport terms Ωd (rotation) and Ad (stretching) cannot be properly defined, because Ω and A are 2×2 matrices but d is a 3D vector. We want to mention that such problem does not apply to simplified liquid crystal system of small molecules [30]. In particular, we refer to recent works [29, 33, 49] for a simplified liquid crystal system in 2D but the director $d : \Omega \times [0, T] \rightarrow S^2$, which is three dimensional (with the constraint $|d| = 1$).*

Plan of the paper. The remaining part of the paper is organized as follows. In Section 2, we discuss the basic energy dissipation law of the system (1.11)–(1.13). In Section 3, for the given energy law, we re-derive the Ericksen–Leslie system via an energy variational approach. In particular, under Parodi’s relation, we show the specific relations between results from Least Action Principle and those from Maximum Dissipation Principle. In Section 4, we prove global well-posedness under large viscosity assumption on μ_4 and the long-time behavior of global solutions. In particular, we show that any global solution will converge to a single steady state as time tends to infinity and provide an estimate on the convergence rate. In Section 5, we prove the well-posedness and stability when the initial velocity is near zero and the initial director is close to a local energy minimizer under Parodi’s relation. In Section 6, we discuss the connection between Parodi’s relation and linear stability of the original Ericksen–Leslie system. In Section 7, the appendix section, we present some detailed calculations needed for the previous sections.

2 Basic Energy Law

Generally speaking, singularities that can be observed for a physical system are those energetically admissible ones (cf. [34]). It has been pointed out that the Ericksen–Leslie system (1.11)–(1.15) obeys some dissipative energy inequality under proper assumptions on those physical coefficients (cf. [32]).

The total energy of the Ericksen–Leslie system (1.11)–(1.15) consists of kinetic and potential energies and it is given by

$$\mathcal{E}(t) = \frac{1}{2}\|v\|^2 + \frac{1}{2}\|\nabla d\|^2 + \int_Q \mathcal{F}(d)dx. \quad (2.1)$$

For the sake of simplicity, we denote the inner product on $L^2(Q)$ (or $\mathbf{L}^2(Q)$, for the corresponding vector space) by (\cdot, \cdot) and the associated norm by $\|\cdot\|$.

By a direct calculation with smooth solutions (v, d) to the system (1.11)–(1.15), we have (cf. [32, Theorem 2.1] for detailed calculations for the corresponding initial boundary value problem)

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) = & - \int_Q \left(\mu_1 |d^T Ad|^2 + \frac{\mu_4}{2} |\nabla v|^2 + (\mu_5 + \mu_6) |Ad|^2 \right) dx \\ & + \lambda_1 \|N\|^2 + (\lambda_2 - \mu_2 - \mu_3)(N, Ad). \end{aligned} \quad (2.2)$$

Here and after, we always assume that

$$\lambda_1 < 0, \quad (2.3)$$

$$\mu_5 + \mu_6 \geq 0, \quad (2.4)$$

$$\mu_1 \geq 0, \quad \mu_4 > 0. \quad (2.5)$$

These assumptions are assumed to provide necessary conditions for the dissipation of the director field [15, 27]. As indicated in [32], the assumption (1.16) guarantees the existence of the Lyapunov-type functional. However, we note that Parodi's relation (1.17) is not necessary in the derivation of (2.2). If (1.17) is employed, i.e., $\lambda_2 = -(\mu_2 + \mu_3)$, we immediately arrive at the energy inequality obtained in [32, Theorem 2.1]. Moreover, if we further assume $\lambda_2 = 0$, it follows from (2.2)–(2.5) that $\mathcal{E}(t)$ is decreasing in time, which is exactly the case studied in [32].

Lemma 2.1 (Basic energy law with Parodi's relation). *Suppose that the assumptions (1.16), (1.17), (2.3), (2.4) and (2.5) are satisfied. In addition, if we assume*

$$\frac{(\lambda_2)^2}{-\lambda_1} \leq \mu_5 + \mu_6, \quad (2.6)$$

then the total energy $\mathcal{E}(t)$ is decreasing in time such that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) = & - \int_Q \left(\mu_1 |d^T Ad|^2 + \frac{\mu_4}{2} |\nabla v|^2 \right) dx + \frac{1}{\lambda_1} \|\Delta d - f(d)\|^2 \\ & - \left(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1} \right) \|Ad\|^2 \\ \leq & 0. \end{aligned} \quad (2.7)$$

Proof. By Parodi's relation (1.17), i.e., $\lambda_2 = -(\mu_2 + \mu_3)$, we infer from the transport equation of d (cf. (1.13)) that

$$\begin{aligned} & \lambda_1 \|N\|^2 + (\lambda_2 - \mu_2 - \mu_3)(N, Ad) \\ \stackrel{(1.17)}{=} & (N, \lambda_1 N + \lambda_2 Ad) + \lambda_2 (N, Ad) \\ = & (N, \lambda_1 N + \lambda_2 Ad) + \left(N + \frac{\lambda_2}{\lambda_1} Ad, \lambda_2 Ad \right) - \frac{(\lambda_2)^2}{\lambda_1} \|Ad\|^2 \\ = & \lambda_1 \left\| N + \frac{\lambda_2}{\lambda_1} Ad \right\|^2 - \frac{(\lambda_2)^2}{\lambda_1} \|Ad\|^2 \\ \stackrel{(1.13)}{=} & \frac{1}{\lambda_1} \|\Delta d - f(d)\|^2 - \frac{(\lambda_2)^2}{\lambda_1} \|Ad\|^2. \end{aligned} \quad (2.8)$$

Inserting the above result into (2.2), we arrive at our conclusion. \square \square

On the contrary, if Parodi's relation (1.17) does not hold, additional assumptions have to be imposed to ensure the dissipation of the total energy.

Lemma 2.2 (Basic energy law without Parodi's relation). *Suppose that (1.16), (2.3), (2.4) and (2.5) are satisfied. If we further assume that*

$$|\lambda_2 - \mu_2 - \mu_3| \leq 2\sqrt{-\lambda_1}\sqrt{\mu_5 + \mu_6}, \quad (2.9)$$

then the following energy inequality holds:

$$\frac{d}{dt}\mathcal{E}(t) \leq - \int_Q \left(\mu_1 |d^T Ad|^2 + \frac{\mu_4}{2} |\nabla v|^2 \right) dx \leq 0. \quad (2.10)$$

Moreover, if

$$|\lambda_2 - \mu_2 - \mu_3| < 2\sqrt{-\lambda_1}\sqrt{\mu_5 + \mu_6}, \quad (2.11)$$

then the dissipation in (2.10) will be stronger in the sense that there exists a small constant $\eta > 0$,

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &\leq - \int_Q \left(\mu_1 |d^T Ad|^2 + \frac{\mu_4}{2} |\nabla v|^2 \right) dx - \eta(\|Ad\|^2 + \|N\|^2) \\ &\leq 0. \end{aligned} \quad (2.12)$$

Proof. The conclusion easily follows from (2.2) and the Cauchy–Schwarz inequality. $\square \quad \square$

3 Energy Variational Approaches

The energy variational approaches (*EnVarA*) provide unified variational frameworks in studying complex fluids with microstructures (cf. [20]). From the energetic point of view, the Ericksen–Leslie system (1.11)–(1.13) exhibits competition between the macroscopic flow field and the microscopic director field, through the coupling between the kinematic transport of the director d by the macroscopic velocity field v and the averaged microscopic effect in the form of induced macroscopic elastic stress on the macroscopic flow field. This contributes to some interesting hydrodynamic and rheological properties of the liquid crystal flows. Based on the basic energy law in Section 2, and due to the special feature of nematic liquid crystal flow such that the molecular orientations are transported and deformed by the flow under parallel transport, we shall develop a formal physical derivation of the induced elastic stress through *EnVarA*. This will provide us with a further understanding of the competition between hydrodynamic kinetic energy and internal elastic energy due to the presence of the orientation field d .

The energetic variational treatment of complex fluids starts with the energy dissipative law for the whole coupled system [20, 50]:

$$\frac{dE^{tot}}{dt} = -\mathcal{D},$$

where $E^{tot} = E^{kinetic} + E^{int}$ is the total energy consisting of the kinetic energy and free energy. Here \mathcal{D} is the dissipation function which is equal to the entropy production of the system in isothermal situations. Following Onsager's linear response assumption, we assume that \mathcal{D} is a linear combination of the squares of various rate functions such as velocity, rate of strain or the material derivative of internal variables (cf. [20, 37–39, 50]). The *EnVarA* combines the maximum dissipation principle (for long time dynamics) and the least action principle, or equivalently, the principle of virtual work (for intrinsic and short time dynamics) into a force balance law that expands the conservation law of momentum to include dissipation (cf. [6, 23]). The least action principle gives us the Hamiltonian (reversible) part of the system related to conservative forces. Meanwhile, the maximum dissipation principle provides the dissipative (irreversible) part of

the system related to dissipative forces. In this way, we can distinguish the conservative and dissipative parts among the induced stress terms.

In the context of basic mechanics, both hydrodynamics and elasticity, the basic variable is the flow map $x(X, t)$ (particle trajectory for any fixed X). Here, X is the original labeling (the Lagrangian coordinate) of the particle, which is also referred to as the material coordinate, while x is the current (Eulerian) coordinate and is also called the reference coordinate. For a given velocity field $v(x, t)$, the flow map is defined by the ordinary differential equations:

$$x_t = v(x(X, t), t), \quad x(X, 0) = X.$$

The deformation tensor \mathbb{F} associated with the flow field is given by

$$\mathbb{F}_{ij} = \frac{\partial x_i}{\partial X_j}.$$

Without ambiguity, we define $\mathbb{F}(x(X, t), t) = \mathbb{F}(X, t)$. Applying the chain rule, we can see that $\mathbb{F}(x, t)$ and $\mathbb{F}^{-T}(x, t)$ satisfy the following transport equations (cf. e.g., [17, 24])

$$\begin{aligned} \mathbb{F}_t + v \cdot \nabla \mathbb{F} &= \nabla v \mathbb{F}, \\ \mathbb{F}_t^{-T} + v \cdot \nabla \mathbb{F}^{-T} &= -\nabla^T v \mathbb{F}^{-T}. \end{aligned}$$

Kinematic transport of the director field d represents the (microscopic) molecules moving in the (macroscopic) flow [24, 44]. For general ellipsoid shaped liquid crystal molecules, the transport of d can be represented by

$$d(x(X, t), t) = \mathbb{E}d_0(X) \quad (3.1)$$

with $d_0(X)$ being the initial configuration. The deformation tensor $\mathbb{E}(x(X, t), t)$ carries all the information of micro structures and configurations. It satisfies the following transport equation whose right-hand side can also be reformulated into a combination of a symmetric part and a skew part: (cf. [21, 34, 44])

$$\begin{aligned} \mathbb{E}_t + v \cdot \nabla \mathbb{E} &= \left[\alpha \nabla v + (1 - \alpha)(-\nabla^T v) \right] \mathbb{E} \\ &= \Omega \mathbb{E} + (2\alpha - 1)A\mathbb{E}. \end{aligned} \quad (3.2)$$

Such solutions are called Jeffrey's orbits (cf. [21]). By the fundamental work of Jeffrey [21], the parameter

$$\eta = 2\alpha - 1 = \frac{r^2 - 1}{r^2 + 1} \in [-1, 1], \quad r \in \mathbb{R}$$

is related to the aspect ratio of the ellipsoids. Recently, we have shown that η can also be related to the slippage between the particles and the flow [44]. In the present case, we see that

$$\alpha = \frac{1}{2} \left(1 - \frac{\lambda_2}{\lambda_1} \right).$$

In what follows, we shall apply *EnVarA* to recover the system (1.11)–(1.13) from the basic energy law under the assumption that both (1.16) and Parodi's relation (1.17) are satisfied. The kinetic energy $E^{kinetic}$ and internal elastic energy E^{int} of the system (1.11)–(1.13) are given by

$$E^{kinetic} = \frac{1}{2} \|v\|^2, \quad E^{int} = E(d) = \frac{1}{2} \|\nabla d\|^2 + \int_Q \mathcal{F}(d) dx.$$

The Legendre transformation yields the action functional \mathbb{A} of the particle trajectories in terms of the flow map $x(X, t)$:

$$\mathbb{A}(x) = \int_0^T (E^{kinetic} - E^{int}) dt,$$

which represents the competition between the kinetic energy and the internal energy. If there is no internal microscopic damping, we deduce the total (pure) transport equation of d from (3.2) such that

$$\begin{aligned}\frac{Dd}{Dt} &= d_t + v \cdot \nabla d - \alpha \nabla v d + (1 - \alpha)(\nabla^T v)d \\ &= d_t + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d \\ &= 0.\end{aligned}\tag{3.3}$$

The least action principle optimizes the action \mathbb{A} with respect to all volume preserving trajectories $x(X, t)$, i.e., $\delta_x \mathbb{A} = 0$, with incompressibility of the fluid $\nabla \cdot v = 0$. Consequently, we obtain the conservative force balance equation of classical Hamiltonian mechanics (see (7.6) for its weak variational form)

$$v_t + v \cdot \nabla v = -\nabla P - \nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \tilde{\sigma},\tag{3.4}$$

where

$$\begin{aligned}\tilde{\sigma} &= -\frac{1}{2} \left(1 - \frac{\lambda_2}{\lambda_1}\right) (\Delta d - f(d)) \otimes d \\ &\quad + \frac{1}{2} \left(1 + \frac{\lambda_2}{\lambda_1}\right) d \otimes (\Delta d - f(d)).\end{aligned}\tag{3.5}$$

Here, the symbol \otimes denotes the usual Kronecker multiplication, namely, $(a \otimes b)_{i,j} = a_i b_j$ for $a, b \in \mathbb{R}^3$ and $1 \leq i, j \leq 3$. We also note that the stress tensor $\tilde{\sigma}$ is not symmetric due to the different coefficients of its two components. Together with (3.3), we recover the conservative (Hamiltonian) part of the full system (1.11)–(1.13) (see Section 7.1 for the detailed calculations).

On the other hand, taking the internal dissipation into account together with the transport equation (3.3), we get

$$\begin{aligned}d_t + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d &= \frac{1}{\lambda_1} \frac{\delta E^{int}}{\delta d} \\ &= -\frac{1}{\lambda_1} (\Delta d - f(d)),\end{aligned}\tag{3.6}$$

which reflects the elastic relaxation dynamics. The dissipation functional \mathcal{D} to the system (1.11)–(1.13) is in terms of the variables A and N (cf. (2.2)) (we remark that our dissipation functional, like in [2], departs from those loosely defined by Onsager in [38]). Under Parodi's relation (1.17) and by (3.6), it can be transformed into the following form (cf. (2.8))

$$\begin{aligned}\mathcal{D} &= \mu_1 \|d^T A d\|^2 + \frac{\mu_4}{2} \|\nabla v\|^2 - \lambda_1 \left\| d_t + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d \right\|^2 \\ &\quad + \left(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1} \right) \|A d\|^2.\end{aligned}\tag{3.7}$$

According to the maximum dissipation principle [37–39], we take $\delta_v (\frac{1}{2} \mathcal{D}) = 0$ (performing variation with respect to the rate function, i.e., the velocity v in Eulerian coordinate) with incompressibility of the fluid $\nabla \cdot v = 0$. This yields the dissipative force balance law equivalent to the conservation of momentum (see Section 7.2 for the detailed calculations):

$$0 = -\nabla P - \nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \sigma,\tag{3.8}$$

where

$$\begin{aligned}\sigma &= \mu_1(d^T Ad)d \otimes d + \mu_2 N \otimes d + \mu_3 d \otimes N + \mu_4 A \\ &\quad + \mu_5 Ad \otimes d + \mu_6 d \otimes Ad,\end{aligned}\tag{3.9}$$

with constants

$$\mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2), \quad \mu_3 = -\frac{1}{2}(\lambda_1 + \lambda_2).$$

Combining (3.8) with (3.6), we recover the dissipative part of the full system (1.11)–(1.13), which stands for the macroscopic long time dynamics.

The Ericksen–Leslie system (1.11)–(1.13) is the hybrid of these two conservative/dissipative systems. Combining the dissipative part derived from maximum dissipation principle (cf. (3.8)) with the conservative part derived from the least action principle (cf. (3.4)), and taking into account the total equation of d (3.6), we recover the full system (1.11)–(1.13).

Remark 3.1. We first observe from (3.5) and (3.6) (i.e., $-\lambda_1 N - \lambda_2 A d = \Delta d - f(d)$) that

$$\tilde{\sigma} = \mu_2 N \otimes d + \mu_3 d \otimes N + \eta_5 A d \otimes d + \eta_6 d \otimes A d,\tag{3.10}$$

with constants

$$\begin{aligned}\mu_2 &= \frac{1}{2}(\lambda_1 - \lambda_2), \quad \mu_3 = -\frac{1}{2}(\lambda_1 + \lambda_2), \\ \eta_5 &= \frac{1}{2} \left(\lambda_2 - \frac{(\lambda_2)^2}{\lambda_1} \right), \quad \eta_6 = -\frac{1}{2} \left(\lambda_2 + \frac{(\lambda_2)^2}{\lambda_1} \right).\end{aligned}\tag{3.11}$$

The interesting fact from the above derivation is that the induced stress terms

$$-\nabla d \odot \nabla d + \mu_2 N \otimes d + \mu_3 d \otimes N + \eta_5 A d \otimes d + \eta_6 d \otimes A d$$

can be derived either by the least action principle (cf. (3.10)) or the maximum dissipation principle (contained in (3.9)). Therefore, they can either be recognized as conservative or dissipative. However, the remaining part in (3.9)

$$\mu_1(d^T Ad)d \otimes d + \mu_4 A + (\mu_5 - \eta_5)A d \otimes d + (\mu_6 - \eta_6)d \otimes A d\tag{3.12}$$

can only be derived by the maximum dissipation principle. This fact indicates that these terms in (3.12) are dissipative. In particular, at the critical value of λ_2 , i.e.,

$$|\lambda_2| = \sqrt{-\lambda_1} \sqrt{\mu_5 + \mu_6},\tag{3.13}$$

the dissipation functional of the system (1.11)–(1.13) is reduced to

$$\mathcal{D} = \mu_1 \|d^T Ad\|^2 + \frac{\mu_4}{2} \|\nabla v\|^2 - \lambda_1 \left\| d_t + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d \right\|^2.$$

It turns out that $\mu_5 = \eta_5$, $\mu_6 = \eta_6$ and the only two dissipative terms are given by those associated with μ_1 and μ_4 . \square

Finally, we look at some special cases of the system (1.11)–(1.13). We assume that (1.16)–(1.17) are satisfied and set

$$\begin{aligned}\mu_1 &= 0, \\ \mu_2 &= \frac{1}{2}(\lambda_1 - \lambda_2), \quad \mu_3 = -\frac{1}{2}(\lambda_1 + \lambda_2),\end{aligned}$$

$$\mu_5 = \frac{1}{2} \left(\lambda_2 - \frac{(\lambda_2)^2}{\lambda_1} \right), \quad \mu_6 = -\frac{1}{2} \left(\lambda_2 + \frac{(\lambda_2)^2}{\lambda_1} \right).$$

Since (3.13) is now satisfied, then the system (1.11)–(1.13) can be reduced to

$$v_t + v \cdot \nabla v + \nabla p = \frac{\mu_4}{2} \Delta v - \nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \sigma, \quad (3.14)$$

$$\nabla \cdot v = 0, \quad (3.15)$$

$$d_t + v \cdot \nabla d - \frac{\mu_2}{\lambda_1} \nabla v d - \frac{\mu_3}{\lambda_1} \nabla^T v d = -\frac{1}{\lambda_1} (\Delta d - f), \quad (3.16)$$

where

$$\sigma = -\frac{\mu_2}{\lambda_1} (\Delta d - f) \otimes d - \frac{\mu_3}{\lambda_1} d \otimes (\Delta d - f). \quad (3.17)$$

Remark 3.2. *The system (3.14)–(3.17) is consistent with these simplified models studied in [16, 34, 44, 48]:*

(1) *The rod-like molecule model:*

$$\mu_2 = \lambda_1 = -\lambda_2, \quad \mu_3 = 0.$$

In this case, the director field d satisfies the kinematic transport relation

$$d(x(X, t), t) = \mathbb{F} d_0(X), \quad \text{where} \quad \dot{\mathbb{F}} = \nabla v \mathbb{F}.$$

(2) *The disc-like molecule model:*

$$\mu_2 = 0, \quad \mu_3 = -\lambda_1 = -\lambda_2.$$

In this case, d satisfies

$$d(x(X, t), t) = \mathbb{F}^{-T} d_0(X), \quad \text{where} \quad \dot{\mathbb{F}}^{-T} = -\nabla^T v \mathbb{F}^{-T}.$$

(3) *The sphere-like molecule model:*

$$\mu_2 = \frac{\lambda_1}{2}, \quad \mu_3 = -\frac{\lambda_1}{2}, \quad \lambda_2 = 0.$$

In this case, d satisfies

$$d(x(X, t), t) = \mathbb{E} d_0(X), \quad \text{where} \quad \dot{\mathbb{E}} = \frac{1}{2} (\nabla v - \nabla^T v) \mathbb{E}.$$

4 Well-posedness and Long-time Behavior for Large Viscosity

μ_4

For any Banach space X , we denote by \mathbf{X} the vector space $(X)^r$, $r \in \mathbb{N}$, endowed with the product norms. We recall the well established functional settings for periodic problems (cf. [45]):

$$\begin{aligned} H_p^m(Q) &= \{u \in H^m(\mathbb{R}^3, \mathbb{R}) \mid u(x + e_i) = u(x)\}, \\ \dot{H}_p^m(Q) &= H_p^m(Q) \cap \left\{ u : \int_Q u(x) dx = 0 \right\}, \\ H &= \{v \in \mathbf{L}_p^2(Q), \nabla \cdot v = 0\}, \quad \text{where } \mathbf{L}_p^2(Q) = \mathbf{H}_p^0(Q), \\ V &= \{v \in \dot{\mathbf{H}}_p^1(Q), \nabla \cdot v = 0\}, \\ V' &= \text{the dual space of } V. \end{aligned}$$

We denote the inner product on $L_p^2(Q)$ (or $\mathbf{L}_p^2(Q)$) as well as H by (\cdot, \cdot) and the associated norm by $\|\cdot\|$. The space $H_p^m(Q)$ will be short-handed by H_p^m and the H^m -inner product ($m \in \mathbb{N}$) can be given by $\langle v, u \rangle_{H^m} = \sum_{|\kappa|=0}^m (D^\kappa v, D^\kappa u)$, where $\kappa = (\kappa_1, \dots, \kappa_n)$ is a multi-index of length $|\kappa| = \sum_{i=1}^n \kappa_i$ and $D^\kappa = \partial_{x_1}^{\kappa_1} \dots \partial_{x_n}^{\kappa_n}$. We denote by C the genetic constant possibly depending on $\lambda'_i s, \mu'_i s, Q, f$ and the initial data. Special dependence will be pointed out explicitly if necessary. Throughout the paper, the Einstein summation convention will be used.

As mentioned in the introduction, we use the Ginzburg–Landau approximation to reduce the order of nonlinearities caused by the constraint $|d| = 1$. We note that either for a highly simplified liquid crystal model (cf. [30]), or for the general Ericksen–Leslie system (1.11)–(1.15) with the artificial assumption $\lambda_2 = 0$ (cf. [32]), a certain type of maximum principle holds for the d -equation, namely, if $|d_0| \leq 1$ then $|d| \leq 1$. This fact still holds for our current periodic settings with the same assumption on λ_2 . Then combining the basic energy law, one can deduce that

$$v \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad (4.1)$$

$$d \in L^\infty(0, T; \mathbf{H}_p^1 \cap \mathbf{L}_p^\infty) \cap L^2(0, T; \mathbf{H}_p^2), \quad (4.2)$$

which is sufficient for the following formulation of weak solutions:

Definition 4.1. *(v, d) is called a weak solution of (1.11)–(1.15) in $Q_T = Q \times (0, T)$ if it satisfies (4.1), (4.2) and for any smooth function $\psi(t)$ with $\psi(T) = 0$ and $\phi(x) \in \mathbf{H}_p^1$, the following weak formulation together with the initial and boundary conditions (1.14) and (1.15) hold:*

$$\begin{aligned} & - \int_0^T (v, \psi_t \phi) dt + \int_0^T (v \cdot \nabla v, \psi \phi) dt \\ = & - (v_0, \phi) \psi(0) + \int_0^T (\nabla d \odot \nabla d, \psi \nabla \phi) dt - \int_0^T (\sigma, \psi \nabla \phi) dt, \end{aligned}$$

where σ is defined in (1.4), and

$$\begin{aligned} & - \int_0^T (d, \psi_t \phi) dt + \int_0^T (v \cdot \nabla d, \psi \phi) dt - \int_0^T (\Omega d, \psi \phi) dt + \frac{\lambda_2}{\lambda_1} \int_0^T (Ad, \psi \phi) dt \\ = & - (d_0, \phi) \psi(0) - \frac{1}{\lambda_1} \int_0^T (\Delta d - f(d), \psi \phi) dt. \end{aligned}$$

With the help of the maximum principle under the assumption $\lambda_2 = 0$, in [32], the authors obtained the existence of weak solutions by applying a semi-Galerkin procedure (cf. [30] for the simplified liquid crystal system). For the more general case considered in the present paper, we no longer assume that $\lambda_2 = 0$. Consequently, the kinetic transport includes the stretching effect that leads to the loss of maximum principle for d . In order to ensure that the extra stress term $\nabla \cdot \sigma$ is well defined in the weak formulation (cf. Definition 4.1), the regularity

$$d \in L^\infty(0, T; \mathbf{L}^\infty)$$

turns out to be essential (we refer to [44] for the discussions on the rod-like molecule liquid crystal model, which is a special case of the general system (1.11)–(1.15)). In the subsequent analysis, we have to confine ourselves to the periodic boundary conditions, which helps us to avoid extra difficulties involving boundary terms when performing integration by parts in the derivation of higher-order energy inequalities.

Finally, we remark that existence of global weak solutions to simplified liquid crystal systems in Remark 3.2 has been obtained in [3] with a suitable set of boundary conditions (i.e., homogeneous Dirichlet boundary condition for v together with the homogeneous Neumann boundary

condition for d). Their argument is based on an appropriate choice of test functions that leads to a suitable weak formulation of the system and thus overcomes difficulties from the stretching effect. Quite recently, the existence of global weak solutions with energy bounds to the general Ericksen–Leslie system (1.11)–(1.13) has been proved in [4] by extending the argument in [3]. Moreover, in [4], under Parodi’s relation, the authors prove the local existence/uniqueness of classical solutions to the general Ericksen–Leslie system (1.11)–(1.13) and establish a Beale–Kato–Majda type blow-up criterion.

4.1 Galerkin approximation

Under periodic settings, one can define a mapping S associated with the Stokes problem: $Su = -\Delta u$ for $u \in D(S) = \{u \in H, Su \in H\} = \dot{\mathbf{H}}_p^2 \cap H$. The operator S can be seen as an unbounded positive linear self-adjoint operator on H . If $D(S)$ is endowed with the norm induced by $\dot{\mathbf{H}}_p^0$, then S becomes an isomorphism from $D(S)$ onto H .

Let $\{\phi_i\}_{i=1}^\infty$ with $\|\phi_i\| = 1$ be the eigenvectors of the Stokes operator S in the periodic case with zero mean,

$$-\Delta \phi_i + \nabla P_i = \kappa_i \phi_i, \quad \nabla \cdot \phi_i = 0 \quad \text{in } Q, \quad \int_Q \phi_i(x) dx = 0,$$

where $P_i \in L^2$ and $0 < \kappa_1 \leq \kappa_2 \leq \dots$ are eigenvalues. The eigenvectors ϕ_i are smooth and the sequence $\{\phi_i\}_{i=1}^\infty$ forms an orthogonal basis of H (cf. [45]). Let

$$P_m : H \rightarrow H_m \doteq \text{span}\{\phi_1, \dots, \phi_m\}, \quad m \in \mathbb{N}.$$

We consider the following (variational) approximate problem:

$$\begin{aligned} &(\partial_t v_m, u_m) + (v_m \cdot \nabla v_m, u_m) \\ &= (\nabla d_m \odot \nabla d_m, \nabla u_m) - (\sigma_m, \nabla u_m), \quad \forall u_m \in H_m, \end{aligned} \quad (4.3)$$

$$N_m + \frac{\lambda_2}{\lambda_1} A_m d_m = -\frac{1}{\lambda_1} \Delta d_m - f(d_m), \quad (4.4)$$

$$v_m(x, 0) = P_m v_0(x), \quad d_m(x, 0) = d_0(x), \quad (4.5)$$

$$v_m(x + e_i, t) = v_m(x, t), \quad d_m(x + e_i, t) = d_m(x, t), \quad (4.6)$$

where

$$\begin{aligned} \Omega_m &= \frac{1}{2}(\nabla v_m - \nabla^T v_m), \quad A_m = \frac{1}{2}(\nabla v_m + \nabla^T v_m), \\ N_m &= \partial_t d_m + (v_m \cdot \nabla) d_m + \Omega_m d_m, \\ \sigma_m &= \mu_1(d_m^T A_m d_m) d_m \otimes d_m + \mu_2 N_m \otimes d_m + \mu_3 d_m \otimes N_m + \mu_4 A_m \\ &\quad + \mu_5 A_m d_m \otimes d_m + \mu_6 d_m \otimes A_m d_m. \end{aligned}$$

We can prove local well-posedness of the approximate problem (4.3)–(4.6) by a similar semi-Galerkin procedure like [44] (see also [30, 32]). Smoothness of the approximate solutions in the interior of $Q_{T_0} = (0, T_0) \times Q$ follows from the regularity theory for parabolic equations and a bootstrap argument (cf. [22, 30]). The uniqueness of smooth solutions can be proved by performing energy estimates on the difference of two different solutions and using Gronwall’s inequality. Since the proof is standard, we omit the details here.

Proposition 4.1. *Suppose that $v_0 \in V$, $d_0 \in \mathbf{H}_p^2$. For any $m > 0$, there is a $T_0 > 0$ depending on v_0 , d_0 and m such that the approximate problem (4.3)–(4.6) admits a unique weak solution (v_m, d_m) satisfying*

$$v_m \in L^\infty(0, T_0; V) \cap L^2(0, T_0; \mathbf{H}_p^2),$$

$$d_m \in L^\infty(0, T_0; \mathbf{H}_p^2) \cap L^2(0, T_0; \mathbf{H}_p^3).$$

Furthermore, (v_m, d_m) is smooth in the interior of Q_{T_0} .

4.2 Uniform *a priori* estimates

In order to prove global existence of solutions to the problem (1.11)–(1.15), we need some uniform estimates that are independent of the approximate parameter m and the local existence time T_0 . These uniform estimates enable us (i) to pass to the limit as $m \rightarrow \infty$ to obtain a weak solution to the system (1.11)–(1.15) in proper spaces; (ii) to extend the local solution to a global one on $[0, +\infty)$. One advantage of the above mentioned semi-Galerkin scheme is that the approximate solutions satisfy the same basic energy law and higher-order differential inequalities as the smooth solutions to the system (1.11)–(1.15). For the sake of simplicity, the following calculations are carried out formally for smooth solutions. They can be justified by using the approximate solutions to (4.3)–(4.6) and then passing to the limit.

The basic energy law plays an important role in the derivation of uniform estimates on $\mathbf{L}^2 \times \mathbf{H}^1$ -norm of (v, d) . According to the discussions in Section 2, we consider the following two cases, in which the basic energy law holds:

- **Case I** (with Parodi's relation): Suppose $\lambda_2 \neq 0$, (1.16), (1.17), (2.3)–(2.6);
- **Case II** (without Parodi's relation): Suppose $\lambda_2 \neq 0$, (1.16), (2.3)–(2.5) and (2.11).

First, we consider **Case I**. It follows from Lemma 2.1 that

$$\frac{d}{dt} \mathcal{E}(t) \leq - \int_Q \mu_1 |d^T A d|^2 dx - \frac{\mu_4}{2} \|\nabla v\|^2 + \frac{1}{\lambda_1} \|\Delta d - f(d)\|^2, \quad \forall t \geq 0.$$

This easily implies the following uniform estimates

$$\|v(t, \cdot)\| \leq C, \quad \|d(t, \cdot)\|_{\mathbf{H}^1} \leq C, \quad \forall t \geq 0, \quad (4.7)$$

$$\int_0^{+\infty} \left(\int_Q \mu_1 |d^T A d|^2 dx + \frac{\mu_4}{2} \|\nabla v\|^2 - \frac{1}{\lambda_1} \|\Delta d - f(d)\|^2 \right) dt \leq C, \quad (4.8)$$

where the constant $C > 0$ depends only on $\|v_0\|$ and $\|d_0\|_{\mathbf{H}^1}$.

As we have mentioned before, the regularity $d \in L^\infty(0, T; \mathbf{L}^\infty)$ is crucial to ensure that the extra stress term $\nabla \cdot \sigma$ can be suitably defined in the weak formulation. Due to the lack of maximum principle for d , an alternative way is to prove higher-order (spatial) regularity of d , e.g., in $L^\infty(0, T; \mathbf{H}^2)$ and use the Sobolev embedding $\mathbf{H}^2 \hookrightarrow \mathbf{L}^\infty$ ($n = 3$). For this purpose, we derive a new type of higher-order energy inequality, which turns out to be useful in the study of global existence of solutions as well as the long-time behavior (cf. [30, 32, 44, 47] for simplified liquid crystal systems).

Lemma 4.1. *Set*

$$\mathcal{A}(t) = \|\nabla v(t)\|^2 + \|\Delta d(t) - f(d(t))\|^2. \quad (4.9)$$

*Let $\underline{\mu}$ be an arbitrary positive constant. We suppose that $\mu_4 \geq \underline{\mu}$. For $n = 3$, under the assumption of **Case I**, the following inequality holds:*

$$\begin{aligned} \frac{d}{dt} \mathcal{A}(t) \leq & - \left(\frac{\mu_4}{2} - C_1 \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}}(t) \right) \|\Delta v\|^2 \\ & + \left(\frac{1}{2\lambda_1} + C_2 \mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}}(t) \right) \|\nabla(\Delta d - f)\|^2 + C_3 \mathcal{A}(t), \end{aligned} \quad (4.10)$$

where

$$\tilde{\mathcal{A}}(t) = \mathcal{A}(t) + 1,$$

C_i ($i = 1, 2, 3$) are constants depending on Q , f , $\|v_0\|$, $\|d_0\|_{\mathbf{H}^1}$, λ_1 , λ_2 , μ_i ($i = 1, 2, 3, 5, 6$) and $\underline{\mu}$.

Proof. Without loss of generality, we assume that $\underline{\mu} = 1$. The argument is valid for arbitrary but fixed $\underline{\mu} > 0$.

Using (1.11)–(1.13) and integration by parts, due to the periodic boundary conditions, we obtain that (see Section 7.3 for detailed computations)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \mathcal{A}(t) + \mu_1 \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + \frac{\mu_4}{2} \|\Delta v\|^2 \\ & + (\mu_5 + \mu_6) \int_Q |d_j \nabla_l A_{ji}|^2 dx - \frac{1}{\lambda_1} \|\nabla(\Delta d - f)\|^2 \\ = & -\mu_1 \int_Q A_{kp} \nabla_l (d_k d_p) d_i d_j \nabla_l A_{ij} dx - \mu_1 \int_Q A_{kp} d_k d_p \nabla_l (d_i d_j) \nabla_l A_{ij} dx \\ & - (\mu_5 + \mu_6) \int_Q \nabla_l d_j d_k A_{ki} \nabla_l A_{ij} dx - (\mu_5 + \mu_6) \int_Q d_j \nabla_l d_k A_{ki} \nabla_l A_{ij} dx \\ & - \int_Q \nabla_l (\Delta d_i - f_i) \Omega_{ij} \nabla_l d_j dx + \int_Q (\Delta d_i - f_i) \nabla_l \Omega_{ij} \nabla_l d_j dx \\ & + 2\lambda_2 \int_Q N_i \nabla_l A_{ij} \nabla_l d_j dx + \lambda_2 (N, A \Delta d) \\ & - \frac{(\lambda_2)^2}{\lambda_1} \int_Q |\nabla_l (A_{ij} d_j)|^2 dx + (\Delta v, v \cdot \nabla v) + \frac{1}{\lambda_1} \int_Q f'(d) |\Delta d - f|^2 dx \\ & - \left(\Delta d - f, f'(d) \left(\Omega d - \frac{\lambda_2}{\lambda_1} A d \right) \right) + 2 \int_Q \nabla_j (\Delta d_i - f_i) \nabla_l v_j \nabla_l d_i dx \\ & - (\Delta d - f, v \cdot \nabla f) \\ \triangleq & I_1 + \dots + I_{14}. \end{aligned} \tag{4.11}$$

In what follows, we estimate the right-hand side of (4.11) term by term.

$$I_1 \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + C \|d\|_{\mathbf{L}^\infty}^2 \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2.$$

By the estimate (4.7), we infer from the Agmon's inequality that

$$\|d\|_{\mathbf{L}^\infty} \leq C(1 + \|\Delta d\|^{\frac{1}{2}}). \tag{4.12}$$

Then from (4.7), (4.12) and the Gagliardo–Nirenberg inequality, we obtain

$$\|\nabla v\|_{\mathbf{L}^3} \leq \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{1}{2}}, \quad \|\nabla v\|_{\mathbf{L}^4} \leq \|\nabla v\|^{\frac{1}{4}} \|\Delta v\|^{\frac{3}{4}}, \tag{4.13}$$

$$\|\nabla d\|_{\mathbf{L}^6} \leq C(\|\Delta d\| + 1), \tag{4.14}$$

$$\|\Delta d\| \leq \|\Delta d - f(d)\| + \|f(d)\| \leq \|\Delta d - f(d)\| + C, \tag{4.15}$$

$$\begin{aligned} \|\nabla \Delta d\| & \leq \|\nabla(\Delta d - f(d))\| + \|\nabla f(d)\| \\ & \leq \|\nabla(\Delta d - f(d))\| + \|f'(d)\|_{\mathbf{L}^\infty} \|\nabla d\| \\ & \leq \|\nabla(\Delta d - f(d))\| + C(1 + \|d\|_{\mathbf{L}^\infty}^2) \end{aligned}$$

$$\begin{aligned}
&\leq \|\nabla(\Delta d - f(d))\| + C(1 + \|\Delta d\|) \\
&\leq \|\nabla(\Delta d - f(d))\| + C(1 + \|\nabla \Delta d\|^{\frac{1}{2}} \|\nabla d\|^{\frac{1}{2}} + \|\nabla d\|) \\
&\leq \|\nabla(\Delta d - f(d))\| + \frac{1}{2} \|\nabla \Delta d\| + C.
\end{aligned} \tag{4.16}$$

As a result, it holds

$$\begin{aligned}
&\|d\|_{\mathbf{L}^\infty}^2 \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2 \\
&\leq C \|\nabla v\| \|\Delta v\| (\|\Delta d - f\|^3 + 1) \\
&\leq \left(\mu_4^{\frac{1}{2}} + \mu_4^{\frac{1}{2}} \|\Delta d - f\|^2 \right) \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 (1 + \|\Delta d - f\|^4) \\
&\leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 \\
&\quad + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 \left(\|\nabla \Delta d\|^{\frac{1}{2}} \|\nabla d\|^{\frac{1}{2}} + \|\nabla d\| + C \right)^4 \\
&\leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 (\|\nabla(\Delta d - f)\|^2 + 1) \\
&\leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C \mathcal{A},
\end{aligned} \tag{4.17}$$

which implies that

$$\begin{aligned}
I_1 &\leq \frac{\mu_1}{4} \int_Q (d_i d_j \nabla_l A_{ij})^2 dx + \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 \\
&\quad + C \mu_4^{-\frac{1}{2}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C \mathcal{A}.
\end{aligned} \tag{4.18}$$

For I_2 , using integration by parts, we obtain

$$\begin{aligned}
I_2 &= \mu_1 \int_Q \nabla_l A_{kp} d_k d_p \nabla_l (d_i d_j) A_{ij} dx + \mu_1 \int_Q A_{kp} \nabla_l (d_k d_p) \nabla_l (d_i d_j) A_{ij} dx \\
&\quad + \mu_1 \int_Q A_{kp} d_k d_p (d_j \Delta d_i + 2 \nabla_l d_i \nabla_l d_j + d_i \Delta d_j) A_{ij} dx \\
&\leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + C \|d\|_{\mathbf{L}^\infty}^2 \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2 \\
&\quad + C \|\nabla v\|_{\mathbf{L}^4}^2 \|\Delta d\| \|d\|_{\mathbf{L}^\infty}^3,
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
&C \|\nabla v\|_{\mathbf{L}^4}^2 \|\Delta d\| \|d\|_{\mathbf{L}^\infty}^3 \\
&\leq C \|\nabla v\|_{\mathbf{L}^4}^{\frac{1}{2}} \|\Delta v\|^{\frac{3}{2}} (\|\Delta d - f\|^{\frac{5}{2}} + 1) \\
&\leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{3}{2}} \|\nabla v\|^2 (1 + \|\Delta d - f\|^4).
\end{aligned} \tag{4.20}$$

Thus, the right-hand side of (4.20) can be estimated exactly as (4.17). Therefore,

$$\begin{aligned}
I_2 &\leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 \\
&\quad + C \mu_4^{-\frac{1}{2}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C \mathcal{A}.
\end{aligned} \tag{4.21}$$

Concerning I_3 and I_4 , we deduce from (4.16) that (using again (4.17) and $\mu_4 \geq 1$)

$$I_3 + I_4 = -(\mu_5 + \mu_6) \int_Q \nabla_l d_j d_k A_{ki} \nabla_l A_{ij} dx$$

$$\begin{aligned}
& -(\mu_5 + \mu_6) \int_Q d_j \nabla_l d_k A_{ki} \nabla_l A_{ij} dx \\
& \leq C \|\Delta v\| \|\nabla v\|_{\mathbf{L}^3} \|\nabla d\|_{\mathbf{L}^6} \|d\|_{\mathbf{L}^\infty} \\
& \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2 \|d\|_{\mathbf{L}^\infty}^2 \\
& \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2 \|d\|_{\mathbf{L}^\infty}^2 \\
& \leq 2\mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}.
\end{aligned} \tag{4.22}$$

Next, for I_5 , I_6 , we have

$$\begin{aligned}
I_5 & \leq C \|\nabla(\Delta d - f)\| \|\nabla v\|_{\mathbf{L}^3} \|\nabla d\|_{\mathbf{L}^6} \\
& \leq C \|\nabla(\Delta d - f)\| \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{1}{2}} (\|\Delta d - f\| + 1) \\
& \leq \mu_4^{\frac{1}{4}} \|\nabla v\| \|\Delta v\| + C \mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}} \|\nabla(\Delta d - f)\|^2 \\
& \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}} \|\nabla(\Delta d - f)\|^2 + C \|\nabla v\|^2,
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
I_6 & \leq \|\nabla \Omega\| \|\Delta d - f\| \|\nabla d\|_{\mathbf{L}^\infty} \\
& \leq C \|\Delta v\| \|\Delta d - f\| (\|\nabla(\Delta d - f)\|^{\frac{3}{4}} + 1) \\
& \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C \mu_4^{-\frac{2}{3}} \|\Delta d - f\|^2 \|\nabla(\Delta d - f)\|^2 + C \|\Delta d - f\|^2 \\
& \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C \mu_4^{-\frac{2}{3}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}.
\end{aligned} \tag{4.24}$$

Using integration by parts and (1.13), we get

$$\begin{aligned}
& I_7 + I_8 + I_9 \\
& = 2\lambda_2 \int_Q N_i \nabla_l A_{ij} \nabla_l d_j dx + \lambda_2 (N, A\Delta d) - \frac{(\lambda_2)^2}{\lambda_1} \int_Q |\nabla_l (A_{ij} d_j)|^2 dx \\
& = \lambda_2 \int_Q N_i \nabla_l A_{ij} \nabla_l d_j dx - \lambda_2 \int_Q \nabla_l N_i A_{ij} \nabla_l d_j dx \\
& \quad - \frac{(\lambda_2)^2}{\lambda_1} \int_Q |A_{ij} \nabla_l d_j|^2 dx - \frac{2(\lambda_2)^2}{\lambda_1} \int_Q \nabla_l A_{ij} d_j A_{ik} \nabla_l d_k dx \\
& \quad - \frac{(\lambda_2)^2}{\lambda_1} \int_Q |\nabla_l A_{ij} d_j|^2 dx \\
& = -\frac{\lambda_2}{\lambda_1} \int_Q (\Delta d_i - f_i) \nabla_l A_{ij} \nabla_l d_j dx + \frac{\lambda_2}{\lambda_1} \int_Q \nabla_l (\Delta d_i - f_i) A_{ij} \nabla_l d_j dx \\
& \quad - \frac{2(\lambda_2)^2}{\lambda_1} \int_Q \nabla_l A_{ij} d_j A_{ij} \nabla_l d_j dx - \frac{(\lambda_2)^2}{\lambda_1} \int_Q |\nabla_l A_{ij} d_j|^2 dx \\
& := K_1 + K_2 + K_3 + K_4.
\end{aligned} \tag{4.25}$$

Similar to (4.22), (4.23) and (4.24), we have

$$\begin{aligned}
& K_1 + K_2 \\
& \leq C \|\Delta v\| \|\Delta d - f\| \|\nabla d\|_{\mathbf{L}^\infty} + C \|\nabla(\Delta d - f)\| \|\nabla v\|_{\mathbf{L}^3} \|\nabla d\|_{\mathbf{L}^6} \\
& \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{4}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}, \\
& K_3 \leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{3}{2}} \|\nabla v\|^2.
\end{aligned}$$

Furthermore, (2.4) and (2.6) indicate that

$$K_4 - (\mu_5 + \mu_6) \int_Q |\nabla_l A_{ij} d_j|^2 dx \leq 0. \quad (4.26)$$

As a result,

$$\begin{aligned} & I_7 + I_8 + I_9 \\ & \leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + (\mu_5 + \mu_6) \int_Q |\nabla_l A_{ij} d_j|^2 dx \\ & \quad + C \mu_4^{-\frac{1}{4}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C \mathcal{A}. \end{aligned}$$

For terms I_{10} , I_{11} and I_{12} , we have

$$\begin{aligned} I_{10} & \leq \|v\|_{\mathbf{L}^4} \|\nabla v\|_{\mathbf{L}^4} \|\Delta v\| \\ & \leq C \|v\|_{\mathbf{L}^4}^{\frac{1}{4}} \|\nabla v\|_{\mathbf{L}^4}^{\frac{3}{4}} \|\Delta v\|_{\mathbf{L}^4}^{\frac{3}{4}} \|\Delta v\| \\ & \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + \mu_4^{\frac{1}{2}} \|\nabla v\|^2 \|\Delta v\|^2 + C \mu_4^{-\frac{7}{2}} \|\nabla v\|^2 \\ & \leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mathcal{A}, \end{aligned} \quad (4.27)$$

$$\begin{aligned} I_{11} & \leq C(\|d\|_{\mathbf{L}^6}^2 + 1) \|\Delta d - f\|_{\mathbf{L}^3}^2 \\ & \leq C \left(\|\Delta d - f\| \|\nabla(\Delta d - f)\| + \|\Delta d - f\|^2 \right) \\ & \leq -\frac{1}{4\lambda_1} \|\nabla(\Delta d - f)\|^2 + C \|\Delta d - f\|^2, \end{aligned} \quad (4.28)$$

and

$$\begin{aligned} I_{12} & = -\left(\Delta d - f, f'(d) \left(\Omega d - \frac{\lambda_2}{\lambda_1} A d \right) \right) \\ & \leq C(\|d\|_{\mathbf{L}^6}^2 + 1) \|d\|_{\mathbf{L}^6} \|\Delta d - f\|_{\mathbf{L}^3} \|\nabla v\|_{\mathbf{L}^6} \\ & \leq C \left(\|\nabla(\Delta d - f)\| + \|\Delta d - f\| \right) \|\nabla v\|_{\mathbf{L}^6}^{\frac{1}{2}} \|\Delta v\|_{\mathbf{L}^6}^{\frac{1}{2}} \\ & \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + \mu_4^{-\frac{1}{4}} \|\nabla(\Delta d - f)\|^2 + C \mathcal{A}. \end{aligned} \quad (4.29)$$

The estimate for I_{13} is exactly the same as (4.23) such that

$$I_{13} \leq \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}} \|\nabla(\Delta d - f)\|^2 + C \|\nabla v\|^2. \quad (4.30)$$

Finally, for I_{14} , we see that

$$\begin{aligned} I_{14} & \leq C \|\Delta d - f\|_{\mathbf{L}^3} \|v\|_{\mathbf{L}^6} \|\nabla f\| \\ & \leq C \|\Delta d - f\|_{\mathbf{L}^3} \|\nabla v\| (1 + \|d\|_{\mathbf{L}^\infty}^2) \|\nabla d\| \\ & \leq C(1 + \|\Delta d - f\|) (\|\nabla(\Delta d - f)\| + \|\Delta d - f\|) \|\nabla v\| \\ & \leq \mu_4^{\frac{1}{2}} \mathcal{A} \|\Delta v\|^2 + \left(-\frac{1}{4\lambda_1} + \mu_4^{-\frac{1}{4}} \right) \|\nabla(\Delta d - f)\|^2 + C(1 + \mu_4^{-\frac{1}{2}}) \mathcal{A}. \end{aligned}$$

Putting all the above estimates together, we arrive at the higher-order differential inequality (4.10). The proof is complete. \square \square

Lemma 4.2. *Under the assumption **Case I**, for any initial data $(v_0, d_0) \in V \times \mathbf{H}^2$, if the viscosity μ_4 is properly large, i.e.,*

$$\mu_4 \geq \mu_4^0(\mu_i, \lambda_1, \lambda_2, v_0, d_0, \underline{\mu}), \quad i = 1, 2, 3, 5, 6,$$

we have

$$\mathcal{A}(t) \leq C, \quad \forall t \geq 0. \quad (4.31)$$

The uniform bound C is a constant depending only on f , Q , $\|v_0\|_V$, $\|d_0\|_{\mathbf{H}^2}$, μ'_s , λ'_s , $\underline{\mu}$.

Proof. It follows from (4.10) that

$$\begin{aligned} & \frac{d}{dt} \tilde{\mathcal{A}}(t) + \left(\frac{\mu_4}{2} - C_1 \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}}(t) \right) \|\Delta v\|^2 \\ & + \left(-\frac{1}{2\lambda_1} - C_2 \mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}}(t) \right) \|\nabla(\Delta d - f)\|^2 \\ & \leq C_3 \tilde{\mathcal{A}}(t). \end{aligned} \quad (4.32)$$

Meanwhile, by (4.8), we have

$$\int_t^{t+1} \tilde{\mathcal{A}}(\tau) d\tau \leq \int_t^{t+1} \mathcal{A}(\tau) d\tau + 1 \leq M, \quad \forall t \geq 0, \quad (4.33)$$

where M is a positive constant depending only on μ'_s (except μ_4), λ'_s , $\|v_0\|$, $\|d_0\|_{\mathbf{H}^1}$. Now we choose μ_4 large enough such that

$$\mu_4^{\frac{1}{2}} \geq 2C_1(\tilde{\mathcal{A}}(0) + 4M + C_3M) + 4\lambda_1^2 C_2^2(\tilde{\mathcal{A}}(0) + 4M + C_3M)^2 + 1. \quad (4.34)$$

Applying a similar argument in [32, Theorem 4.3] (cf. also [30, 48]), we can see that $\tilde{\mathcal{A}}(t)$ is uniformly bounded for all $t \geq 0$ and satisfies

$$\frac{\mu_4}{2} - C_1 \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}}(t) \geq 0, \quad \frac{1}{-2\lambda_1} - C_2 \mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}}(t) \geq 0, \quad \forall t \geq 0. \quad (4.35)$$

The proof is complete. □

Next, we briefly discuss **Case II**.

Corollary 4.1. *For $n = 3$, under the assumption **Case II**, the inequality (4.10) still holds.*

Proof. If Parodi's relation (1.17) does not hold, i.e., $\lambda_2 + (\mu_2 + \mu_3) \neq 0$, then in the derivation of $\frac{d}{dt} \mathcal{A}(t)$ (see Section 7.3), the first term on the right-hand side of (7.19) does not cancel with the first term on the right-hand side of (7.16). Consequently, there is one extra term:

$$(\lambda_2 + \mu_2 + \mu_3) \int_Q d_j N_i \Delta A_{ij} dx.$$

Besides, since we no longer have (2.6) in **Case II**, we have to re-investigate the left-hand side of (4.26). Using the d equation (1.13) and integration by parts, we get

$$\begin{aligned} & (\lambda_2 + \mu_2 + \mu_3) \int_Q d_j N_i \Delta A_{ij} dx \\ & = \frac{\lambda_2 + \mu_2 + \mu_3}{\lambda_1} \int_Q d_j \nabla_l (\Delta d_i - f_i) \nabla_l A_{ij} dx \\ & \quad + \frac{\lambda_2 + \mu_2 + \mu_3}{\lambda_1} \int_Q \nabla_l d_j (\Delta d_i - f_i) \nabla_l A_{ij} dx \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda_2(\lambda_2 + \mu_2 + \mu_3)}{\lambda_1} \int_Q |d_j \nabla_l A_{ij}|^2 dx \\
& + \frac{\lambda_2(\lambda_2 + \mu_2 + \mu_3)}{\lambda_1} \int_Q \nabla_l d_j A_{ik} d_k \nabla_l A_{ij} dx \\
& + \frac{\lambda_2(\lambda_2 + \mu_2 + \mu_3)}{\lambda_1} \int_Q d_j A_{ik} \nabla_l d_k \nabla_l A_{ij} dx.
\end{aligned} \tag{4.36}$$

We now estimate the right-hand side of (4.36). For the first term, we have

$$\begin{aligned}
& \frac{\lambda_2 + \mu_2 + \mu_3}{\lambda_1} \int_Q d_j \nabla_l (\Delta d_i - f_i) \nabla_l A_{ij} dx \\
& \leq C \|d\|_{\mathbf{L}^\infty} \|\nabla(\Delta d - f)\| \|\Delta v\| \\
& \leq C (\|\Delta d - f\|^{\frac{1}{2}} + 1) \|\nabla(\Delta d - f)\| \|\Delta v\| \\
& \leq \mu_4^{\frac{1}{4}} (1 + \|\Delta d - f\|) \|\Delta v\|^2 + \frac{C}{\mu_4^{\frac{1}{4}}} \|\nabla(\Delta d - f)\|^2 \\
& \leq \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{4}} \|\nabla(\Delta d - f)\|^2.
\end{aligned} \tag{4.37}$$

The second term can be estimated as (4.24), while the fourth and fifth terms are similar to (4.22). Finally, concerning the third term and the two terms on the left-hand side of (4.26), we infer from (1.16) and (2.11) that

$$\begin{aligned}
& \frac{\lambda_2(\lambda_2 + \mu_2 + \mu_3)}{\lambda_1} - \frac{(\lambda_2)^2}{\lambda_1} - (\mu_5 + \mu_6) \\
& = -\frac{1}{\lambda_1} [\lambda_1(\mu_5 + \mu_6) - \lambda_2(\mu_2 + \mu_3)] \\
& < -\frac{1}{\lambda_1} \left[-\frac{1}{2}(\lambda_2 - \mu_2 - \mu_3)^2 - \lambda_2(\mu_2 + \mu_3) \right] \\
& = \frac{1}{2\lambda_1} [(\lambda_2)^2 + (\mu_2 + \mu_3)^2] \leq 0,
\end{aligned}$$

which yields

$$\left[\frac{\lambda_2(\lambda_2 + \mu_2 + \mu_3)}{\lambda_1} - \frac{(\lambda_2)^2}{\lambda_1} - (\mu_5 + \mu_6) \right] \int_Q |d_j \nabla_l A_{ij}|^2 dx \leq 0.$$

Combining the other estimates in the proof of Lemma 4.1, we obtain the inequality (4.10) under assumption **Case II**. \square \square

Corollary 4.2. *Under the assumption **Case II**, for any initial data $(v_0, d_0) \in V \times \mathbf{H}_p^2$, if the viscosity μ_4 is properly large, i.e.,*

$$\mu_4 \geq \mu_4^0(\mu_i, \lambda_1, \lambda_2, v_0, d_0, \underline{\mu}), \quad i = 1, 2, 3, 5, 6,$$

we have $\mathcal{A}(t) \leq C$ for $t \geq 0$ with C being a constant depending only on f , Q , $\|v_0\|_V$, $\|d_0\|_{\mathbf{H}^2}$, μ'_s , λ'_s and $\underline{\mu}$.

4.3 Global existence and uniqueness

In both **Case I** and **Case II**, the uniform estimates we have obtained in Section 4.2 are independent of the approximation parameter m and time t . This indicates that for both cases, (v_m, d_m) is a global solution to the approximate problem (4.3)–(4.5):

$$v_m \in L^\infty(0, +\infty; V) \cap L_{loc}^2(0, +\infty; \mathbf{H}_p^2),$$

$$d_m \in L^\infty(0, +\infty; \mathbf{H}_p^2) \cap L_{loc}^2(0, +\infty; \mathbf{H}_p^3),$$

which further implies that

$$\partial_t v_m \in L_{loc}^2(0, +\infty; \mathbf{L}_p^2), \quad \partial_t d_m \in L_{loc}^2(0, +\infty; \mathbf{H}_p^1).$$

The uniform estimates enable us to pass to the limit for (v_m, d_m) as $m \rightarrow \infty$. By a similar argument to [30, 44], we can show that there exist a pair of limit functions (v, d) satisfying

$$v \in L^\infty(0, \infty; V) \cap L_{loc}^2(0, +\infty; \mathbf{H}_p^2), \quad (4.38)$$

$$d \in L^\infty(0, +\infty; \mathbf{H}_p^2) \cap L_{loc}^2(0, +\infty; \mathbf{H}_p^3), \quad (4.39)$$

such that (v, d) is a weak solution of the system (1.11)–(1.15). A bootstrap argument based on Serrin's result [42] and Sobolev embedding theorems leads to the existence of classical solutions. The uniqueness of solutions to the problem (1.11)–(1.15) with regularity (4.38)–(4.39) can be proved as in [48, Lemma 2.2].

In summary, we have

Theorem 4.1 (Global well-posedness). *Let $n = 3$. We assume that either the conditions in **Case I** or in **Case II** are satisfied. For any $(v_0, d_0) \in V \times \mathbf{H}_p^2$, under the large viscosity assumption*

$$\mu_4 \geq \mu_4^0(\mu_i, \lambda_1, \lambda_2, v_0, d_0, \underline{\mu}), \quad i = 1, 2, 3, 5, 6,$$

the problem (1.11)–(1.15) admits a unique global solution that satisfies (4.38)–(4.39).

Besides, we have the following continuous dependence on the initial data:

Lemma 4.3. *Suppose that the assumptions in Theorem 4.1 are satisfied. (v_i, d_i) ($i = 1, 2$) are global solutions to the problem (1.11)–(1.15) corresponding to initial data $(v_{0i}, d_{0i}) \in V \times \mathbf{H}_p^2$ ($i = 1, 2$). Then for any $t \in [0, T]$, we have*

$$\begin{aligned} & \| (v_1 - v_2)(t) \|^2 + \| (d_1 - d_2)(t) \|_{\mathbf{H}^1}^2 \\ & + \int_0^t \left(\frac{\mu_4}{2} \| \nabla (v_1 - v_2)(\tau) \|^2 + \| \Delta (d_1 - d_2)(\tau) \|^2 \right) d\tau \\ & \leq 2e^{Ct} (\| v_{01} - v_{02} \|^2 + \| d_{01} - d_{02} \|_{\mathbf{H}^1}^2), \end{aligned}$$

where C is a constant depending on $\| v_{0i} \|_V, \| d_{0i} \|_{\mathbf{H}^2}, \mu's, \lambda's$ but not on t .

Remark 4.1. *If in addition, we assume either*

$$(i) \quad \mu_1 = 0, \lambda_2 \neq 0, \quad \text{or} \quad (ii) \quad \mu_1 \geq 0, \lambda_2 = 0,$$

the same result holds true in 2D without the largeness assumption on μ_4 . In case (i), we note that the nonlinearity of the highest-order vanishes. In particular, this applies to the system (3.14)–(3.16), which is a simplified version of the general Ericksen–Leslie model (cf. [44] for the liquid crystal system with rod-like molecules and [3, 16, 48] with general ellipsoid shape). On the other hand, in case (ii), one can apply the maximum principle for d to obtain its \mathbf{L}^∞ -bound, which makes the proof much easier (cf. [32]). \square

4.4 Long-time behavior: convergence to equilibrium

Now we briefly discuss the long-time behavior of the global solution (v, d) obtained in Theorem 4.1. First, we have the following decay property:

Lemma 4.4. *For the global solutions obtained in Theorem 4.1, we have*

$$\lim_{t \rightarrow +\infty} (\|v(t)\|_V + \|-\Delta d(t) + f(d(t))\|) = 0. \quad (4.40)$$

Proof. We only consider **Case I** and the proof for **Case II** is similar. From the basic energy law (2.7), we see that $\mathcal{A}(t) \in L^1(0, +\infty)$. On the other hand, (4.10) together with (4.31) and (4.35) implies that $\frac{d}{dt}\mathcal{A}(t) \leq C$. As a consequence,

$$\lim_{t \rightarrow +\infty} \mathcal{A}(t) = 0.$$

The proof is complete. □

It easily follows from Lemma 4.4 that

Proposition 4.2. *Suppose that the assumptions in Theorem 4.1 are satisfied. The ω -limit set of $(v_0, d_0) \in V \times \mathbf{H}_p^2$ denoted by $\omega(v_0, d_0)$ is a non-empty bounded connected subset in $V \times \mathbf{H}_p^2$, which is also compact in $\mathbf{L}^2 \times \mathbf{H}_p^1$. Besides, we have*

$$\omega(v_0, d_0) \in \mathcal{S} := \{(0, d) : -\Delta d + f(d) = 0, \text{ in } Q, d(x + e_i) = d(x) \text{ on } \partial Q\}.$$

Therefore, all asymptotic limit points of the system (1.11)–(1.13) satisfy the following reduced stationary problem

$$v_\infty = 0, \quad (4.41)$$

$$\nabla P_\infty + \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) = -\nabla d_\infty \cdot \Delta d_\infty, \quad (4.42)$$

$$-\Delta d_\infty + f(d_\infty) = 0, \quad (4.43)$$

$$d_\infty(x) = d_\infty(x + e_i), \quad x \in \partial Q, \quad (4.44)$$

where in (4.42), we used the well-known fact that (cf. [30])

$$\nabla \cdot (\nabla d_\infty \odot \nabla d_\infty) = \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) + \nabla d_\infty \cdot \Delta d_\infty.$$

(4.42) is a constraint equation for d_∞ . Since $\mathcal{F}'(d) = f(d)$, if d_∞ is a solution to (4.43), then (4.42) is automatically satisfied because all gradients can be absorbed into the pressure.

We have already proved that the velocity field v decays to zero in V as $t \nearrow +\infty$ (cf. Lemma 4.4). On the other hand, we can only conclude *sequential convergence* for d from compactness of the trajectory: for any unbounded sequence $\{t_j\}$, there exist a subsequence $\{t_n\} \nearrow +\infty$ such that

$$\lim_{t_n \rightarrow +\infty} \|d(t_n) - d_\infty\|_{\mathbf{H}^1} = 0, \quad (4.45)$$

where d_∞ satisfies (4.43)–(4.44). The convergence of d for the whole time sequence is non-trivial because in general we cannot expect the uniqueness of critical points of $E(d)$. In the present case, under the periodic boundary conditions, one may see that the dimension of the set of stationary solutions is at least n . This is because a shift in each variable may give another steady state. The convergence of d to a single equilibrium can be achieved by using the well-known Łojasiewicz–Simon approach (cf. L. Simon [43]). We refer to [19] and the references therein for various generalizations and applications. To this end, we introduce a suitable Łojasiewicz–Simon type inequality in the periodic setting (cf. e.g., [19]).

Lemma 4.5 (Łojasiewicz–Simon inequality). *Let ψ be a critical point of the functional*

$$E(d) = \frac{1}{2} \|\nabla d\|^2 + \int_Q \mathcal{F}(d) dx.$$

Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on ψ such that for any $d \in \mathbf{H}_p^1$ satisfying $\|d - \psi\|_{\mathbf{H}^1} < \beta$, it holds

$$\| -\Delta d + f(d) \|_{(\mathbf{H}_p^1)'} \geq |E(d) - E(\psi)|^{1-\theta}, \quad (4.46)$$

where $(\mathbf{H}_p^1)'$ is the dual space of \mathbf{H}_p^1 .

Then we have the following convergence result:

Theorem 4.2 (Convergence to equilibrium). *Under the assumptions of Theorem 4.1, the global solution (v, d) has the following property:*

$$\|v(t)\|_V + \|d(t) - d_\infty\|_{\mathbf{H}^2} \leq C(1+t)^{-\frac{\theta}{(1-2\theta)}}, \quad \forall t \geq 0, \quad (4.47)$$

where d_∞ is a solution to (4.43)–(4.44), C is a constant depending on v_0 , d_0 , f , Q , $\mu'_i s$, $\lambda'_i s$, d_∞ and the constant $\theta \in (0, \frac{1}{2})$ depends on d_∞ (called Łojasiewicz exponent, cf. Lemma 4.5).

Based on Lemma 4.5, the basic energy law deduced in Section 2 (cf. (2.7) or (2.12)) and the higher-order energy inequality (cf. Lemma 4.1 or Corollary 4.1), we can prove Theorem 4.2 following the procedure in [48, Section 3.2, 3.3] with minor modifications. In order not to make the paper too lengthy, we leave the details to interested readers.

5 Well-posedness and Nonlinear Stability under Parodi's Relation

The results obtained in Section 4 indicate that for both **Case I** (with Parodi's relation) and **Case II** (without Parodi's relation), global well-posedness of the Ericksen–Leslie system can be obtained provided that the viscosity μ_4 is properly large. Recall the Navier–Stokes equations in 3D (with periodic boundary conditions and $v_0 \in H$), we can easily derive

$$\frac{d}{dt} \|\nabla v\|^2 + \left(\frac{1}{2} \mu_4 - \mu_4^{\frac{1}{2}} \|\nabla v\|^2 \right) \|\Delta v\|^2 \leq C \mu_4^{-\frac{11}{2}} \|\nabla v\|^2,$$

which implies that the large viscosity assumption is equivalent to small initial data assumption on v in \mathbf{H}^1 -norm. However, this is not the case for the Ericksen–Leslie system (1.11)–(1.15) due to its much more complicated structure (cf. (4.34)). Actually, we do not have the large viscosity/small initial data alternative relation even for those simplified liquid crystal systems [30, 44].

In this section, we show that Parodi's relation (1.17) plays an important role in the well-posedness and stability of the system (1.11)–(1.15), if no additional requirement is imposed on the viscosity μ_4 . In particular, under those assumptions in **Case I**, we are able to prove a suitable higher-order energy inequality that yields the local well-posedness and furthermore, the global existence result provided that the initial velocity v_0 is near zero and the initial director d_0 is close to a *local* minimizer d^* of the elastic energy

$$E(d) = \frac{1}{2} \|\nabla d\|^2 + \int_Q \mathcal{F}(d) dx. \quad (5.1)$$

Besides, we are able to show the Lyapunov stability of *local* energy minimizers of $E(d)$. This implies that Parodi's relation (1.17) serves as a sufficient condition for nonlinear stability of the Ericksen–Leslie system (1.11)–(1.15) from the mathematical point of view.

5.1 Higher-order energy inequality and local well-posedness

Lemma 5.1. *Let $n = 3$. Suppose that the conditions in **Case I** are satisfied. Then the following higher-order energy inequality holds:*

$$\begin{aligned} & \frac{d}{dt} \mathcal{A}(t) + \frac{\mu_1}{2} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + \frac{\mu_4}{8} \|\Delta v\|^2 - \frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 \\ & \leq C_*(\mathcal{A}^6(t) + \mathcal{A}(t)), \end{aligned} \quad (5.2)$$

where C_* is a constant that only depends on μ 's, λ 's, $\|v_0\|$ and $\|d_0\|_{\mathbf{H}^1}$.

Proof. First, from the basic energy law (2.7) we still have the uniform estimates on $\|v(t)\|$ and $\|d(t)\|_{\mathbf{H}^1}$ (cf. (4.7)). Moreover, estimates (4.12)–(4.16) are still valid. Next, we re-estimate the terms I_1, \dots, I_{14} on the right-hand side of (4.11).

$$\begin{aligned} I_1 & \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + C \|d\|_{\mathbf{L}^\infty}^2 \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2 \\ & \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + C (\|\Delta d - f\|^3 + 1) \|\nabla v\| \|\Delta v\| \\ & \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + \frac{\mu_4}{32} \|\Delta v\|^2 \\ & \quad + C (\|\nabla v\|^2 \|\Delta d - f\|^6 + \|\nabla v\|^2) \\ & \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + \frac{\mu_4}{32} \|\Delta v\|^2 + C \mathcal{A}^4 + C \mathcal{A}, \end{aligned} \quad (5.3)$$

$$\begin{aligned} I_2 & \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + C \|d\|_{\mathbf{L}^\infty}^2 \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2 \\ & \quad + C \|d\|_{\mathbf{L}^\infty}^3 \|\nabla v\|_{\mathbf{L}^4}^2 \|\Delta d\| \\ & \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + \frac{\mu_4}{32} \|\Delta v\|^2 + C \mathcal{A}^4 + C \mathcal{A} \\ & \quad + C (\|\Delta d - f\|^{\frac{5}{2}} + 1) \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{3}{2}} \\ & \leq \frac{\mu_1}{4} \int_Q (d_k d_p \nabla_l A_{kp})^2 dx + \frac{\mu_4}{16} \|\Delta v\|^2 + C \mathcal{A}^6 + C \mathcal{A}, \end{aligned}$$

$$\begin{aligned} I_3 + I_4 & \leq C \|\Delta v\| \|\nabla v\|_{\mathbf{L}^3} \|\nabla d\|_{\mathbf{L}^6} \|d\|_{\mathbf{L}^\infty} \\ & \leq \frac{\mu_4}{64} \|\Delta v\|^2 + C \|\nabla v\|_{\mathbf{L}^3}^2 \|\nabla d\|_{\mathbf{L}^6}^2 \|d\|_{\mathbf{L}^\infty}^2 \\ & \leq \frac{\mu_4}{32} \|\Delta v\|^2 + C \mathcal{A}^4 + C \mathcal{A}, \end{aligned} \quad (5.4)$$

$$\begin{aligned} I_5 & \leq C \|\nabla(\Delta d - f)\| \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{1}{2}} (\|\Delta d - f\| + 1) \\ & \leq \frac{\mu_4}{32} \|\Delta v\|^2 - \frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C \|\nabla v\|^2 (\|\Delta d - f\|^4 + 1) \\ & \leq \frac{\mu_4}{32} \|\Delta v\|^2 - \frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C \mathcal{A}^3 + C \mathcal{A}, \end{aligned} \quad (5.5)$$

$$I_6 \leq C \|\Delta v\| \|\Delta d - f\| (\|\nabla(\Delta d - f)\|^{\frac{3}{4}} + 1)$$

$$\leq \frac{\mu_4}{32} \|\Delta v\|^2 - \frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^4 + C\mathcal{A}. \quad (5.6)$$

For the terms K_1, \dots, K_4 in (4.25), we still have (4.26). By a similar argument to (5.5)–(5.6) we get

$$\begin{aligned} K_1 &\leq \frac{\mu_4}{32} \|\Delta v\|^2 - \frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^4 + C\mathcal{A}, \\ K_2 &\leq \frac{\mu_4}{32} \|\Delta v\|^2 - \frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^3 + C\mathcal{A}. \end{aligned}$$

Using integration by parts, we can see that

$$\begin{aligned} K_3 &= -\frac{2(\lambda_2)^2}{\lambda_1} \int_Q \nabla_l A_{ij} d_j A_{ij} \nabla_l d_j dx \\ &= \frac{2(\lambda_2)^2}{\lambda_1} \int_Q |A_{ij} \nabla_l d_j|^2 dx - K_3 + \frac{2(\lambda_2)^2}{\lambda_1} \int_Q A_{ij} d_j A_{ij} \Delta d_j dx, \end{aligned}$$

which together with similar estimates in (5.4) yields that

$$\begin{aligned} K_3 &= \frac{(\lambda_2)^2}{\lambda_1} \int_Q |A_{ij} \nabla_l d_j|^2 dx + \frac{(\lambda_2)^2}{\lambda_1} \int_Q A_{ij} d_j A_{ij} \Delta d_j dx \\ &\leq \frac{\mu_4}{32} \|\Delta v\|^2 + C\mathcal{A}^4 + C\mathcal{A}. \end{aligned}$$

Hence,

$$\begin{aligned} &I_7 + I_8 + I_9 \\ &\leq \frac{3\mu_4}{32} \|\Delta v\|^2 - \frac{1}{4\lambda_1} \|\nabla(\Delta d - f)\|^2 + (\mu_5 + \mu_6) \int_Q |\nabla_l A_{ij} d_j|^2 dx \\ &\quad + C\mathcal{A}^4 + C\mathcal{A}. \end{aligned}$$

The remaining terms can be estimated in a straightforward way.

$$\begin{aligned} I_{10} &\leq |(\Delta v, v \cdot \nabla v)| \leq C \|\Delta v\|^{\frac{7}{4}} \|\nabla v\| \\ &\leq \frac{\mu_4}{32} \|\Delta v\|^2 + C \|\nabla v\|^8, \end{aligned}$$

$$\begin{aligned} I_{11} &\leq C(\|d\|_{\mathbf{L}^6}^2 + 1) \|\Delta d - f\|_{\mathbf{L}^3}^2 \\ &\leq -\frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}, \end{aligned}$$

$$\begin{aligned} I_{12} &\leq C \|f'(d)d\| \|\Delta d - f\|_{\mathbf{L}^6} \|\nabla v\|_{\mathbf{L}^3} \\ &\leq C \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{1}{2}} \left(\|\nabla(\Delta d - f)\| + \|\Delta d - f\| \right) \\ &\leq \frac{\mu_4}{32} \|\Delta v\|^2 - \frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}. \end{aligned}$$

The estimate of I_{13} is similar to (5.5):

$$I_{13} \leq -\frac{1}{8\lambda_1} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^3 + C\mathcal{A}.$$

For the last term I_{14} , we have

$$\begin{aligned}
I_{14} &\leq C\|\Delta d - f\|_{\mathbf{L}^3}\|v\|_{\mathbf{L}^6}\|\nabla f\| \\
&\leq C(1 + \|\Delta d - f\|)(\|\nabla(\Delta d - f)\| + \|\Delta d - f\|)\|\nabla v\| \\
&\leq -\frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^2 + C\mathcal{A}.
\end{aligned}$$

Collecting all the estimates above, we can conclude the higher-order differential inequality (5.2). The proof is complete. \square

The following local well-posedness result is a direct consequence of the higher-order energy inequality (5.2):

Theorem 5.1 (Local well-posedness). *Let $n = 3$. Suppose that the conditions in **Case I** are satisfied. For any $(v_0, d_0) \in V \times \mathbf{H}_p^2$, there exists a $T^* > 0$ such that the problem (1.11)–(1.15) admits a unique local solution satisfying*

$$v \in L^\infty(0, T^*; V) \cap L^2(0, T^*; \mathbf{H}_p^2), \quad d \in L^\infty(0, T^*; \mathbf{H}_p^2) \cap L^2(0, T^*; \mathbf{H}_p^3).$$

Remark 5.1. Unfortunately, we are not able to prove a corresponding local well-posedness result under the assumptions in **Case II** where Parodi's relation (1.17) is not satisfied. In this case the higher-order energy inequality (5.2) is not available any longer. One obvious difficulty is that we lose control of some higher-order nonlinearities that will vanish due to specific cancellations under Parodi's relation (see, e.g., (4.37)). \square

5.2 Near local minimizers: well-posedness and nonlinear stability

Based on Lemma 5.1, one can easily deduce the following property:

Proposition 5.1. *Suppose that the assumptions in **Case I** are satisfied. For any $(v_0, d_0) \in V \times \mathbf{H}_p^2$, if*

$$\|\nabla v\|^2(0) + \|\Delta d - f(d)\|^2(0) \leq R, \tag{5.7}$$

where $R > 0$ is a constant, there exists a positive constant ε_0 depending on μ' 's, λ' 's, $\|v_0\|$, $\|d_0\|_{\mathbf{H}^1}$, f , Q and R , such that the following property holds: for the (unique) local solution (v, d) of the system (1.11)–(1.15) which exists on $[0, T^*]$, if

$$\mathcal{E}(t) \geq \mathcal{E}(0) - \varepsilon_0, \quad \forall t \in [0, T^*],$$

then the local solution (v, d) can be extended beyond T^* .

Proof. We consider the following initial value problem of an ordinary differential equation:

$$\frac{d}{dt}Y(t) = C_*(Y(t)^6 + Y(t)), \quad Y(0) = R \geq \mathcal{A}(0). \tag{5.8}$$

We denote by $I = [0, T_{max})$ the maximal existence interval of $Y(t)$ such that

$$\lim_{t \rightarrow T_{max}^-} Y(t) = +\infty.$$

It follows from the comparison principle that for any $t \in I$, $0 \leq \mathcal{A}(t) \leq Y(t)$. Consequently, $\mathcal{A}(t)$ exists on I . We note that T_{max} is determined by $Y(0) = R$ and C_* such that $T_{max} = T_{max}(R, C_*)$ is increasing as R decreases. Taking

$$t_0 = \frac{3}{4}T_{max}(R, C_*) > 0,$$

then we have

$$0 \leq \mathcal{A}(t) \leq Y(t) \leq K, \quad \forall t \in [0, t_0], \quad (5.9)$$

where K is a constant that only depends on R, C_*, t_0 . This fact combined with the Galerkin approximate scheme in Section 4.1 leads to the local existence of a unique solution to the system (1.11)–(1.15) at least on $[0, t_0]$. (This indeed provides a proof of Theorem 5.1.)

The above argument suggests that the existing time $T^* \geq t_0$. Now if $\mathcal{E}(t) \geq \mathcal{E}(0) - \varepsilon_0$ for all $t \in [0, T^*]$, we infer from Lemma 2.1 that

$$\int_0^{T^*} \int_Q \left(\frac{\mu_4}{2} |\nabla v(t)|^2 - \frac{1}{\lambda_1} |\Delta d(t) - f(d(t))|^2 \right) dx dt \leq \varepsilon_0.$$

Hence, there exists a $t_* \in [T^* - \frac{t_0}{3}, T^*]$ such that

$$\|\nabla v(t_*)\|^2 + \|\Delta d(t_*) - f(d(t_*))\|^2 \leq \max \left\{ \frac{2}{\mu_4}, -\lambda_1 \right\} \frac{3\varepsilon_0}{t_0}.$$

Choosing $\varepsilon_0 > 0$ such that

$$\max \left\{ \frac{2}{\mu_4}, -\lambda_1 \right\} \frac{3\varepsilon_0}{t_0} = R, \quad (5.10)$$

we have $\mathcal{A}(t_*) \leq R$. Taking t_* as the initial time and $Y(t_*) = R$ in (5.8), we infer from the above argument that $Y(t)$ (and thus $\mathcal{A}(t)$) is uniformly bounded at least on $[0, t_* + t_0] \supset [0, T^* + \frac{2}{3}t_0]$. Thus, we can extend the local solution (v, d) from $[0, T^*]$ to $[0, T^* + \frac{2}{3}t_0]$. The proof is complete. \square

Remark 5.2. Proposition 5.1 implies that, for the local solution (v, d) of (1.11)–(1.15), if the total energy $\mathcal{E}(t)$ does not drop too much on its existence interval $[0, T^*]$, then it can be extended beyond T^* . We note that stronger results have been obtained in [30, 32] for simplified liquid crystal systems. In those cases, global existence of weak solutions can be proved and the total energy $\mathcal{E}(t)$ is well-defined on $[0, +\infty)$. Then one can show the alternative relation: either there exists a $T < +\infty$ such that $\mathcal{E}(T) < \mathcal{E}(0) - \varepsilon_0$ or the system admits a (unique) global strong solution. \square

Remark 5.3. It is easy to verify that the above hypothesis on the changing rate of $\mathcal{E}(t)$ can be fulfilled, if the initial velocity v_0 is near zero and the initial molecule director d_0 is close to an absolute minimizer of the elastic energy $E(d)$ (for instance, a constant vector with unit length). We refer to [30, 32, 47] for the cases of the simplified liquid crystal system. The same result holds for our current general case if the same assumption is imposed. \square

The assumption that the initial director d_0 is close to an *absolute* energy minimizer can indeed be improved. Under Parodi's relation (1.17), we can show much stronger result that if v_0 is near zero and d_0 is close to a *local* minimizer of $E(d)$, then the total energy \mathcal{E} will never drop too much. Actually, we shall see below that the global solution will stay close to the given minimizer for all time (i.e., Lyapunov stability) and $\mathcal{E}(t)$ will converge to the same energy level of the *local* minimizer. This generalized result also applies to all those simplified Ericksen–Leslie systems considered in the literature [30, 32, 44, 47, 48].

Definition 5.1. $d^* \in \mathbf{H}_p^1$ is called a local minimizer of $E(d)$, if there exists $\sigma > 0$, such that for any $d \in \mathbf{H}_p^1$ satisfying $\|d - d^*\|_{\mathbf{H}^1} \leq \sigma$, it holds $E(d) \geq E(d^*)$.

Remark 5.4. Since any minimizer of $E(d)$ is also a critical point of $E(d)$, it satisfies the Euler–Lagrange equation

$$-\Delta d + f(d) = 0, \quad x \in Q, \quad d(x) = d(x + e_i), \quad x \in \partial Q. \quad (5.11)$$

From the elliptic regularity theory and bootstrap argument, one can easily see that if the solution $d \in \mathbf{H}_p^1$, then d is smooth. \square

Next, we state the main result of this section:

Theorem 5.2. *Suppose that $n = 3$ and the conditions in **Case I** are satisfied. Let $d^* \in \mathbf{H}_p^2$ be a local minimizer of $E(d)$. There exist positive constants σ_1, σ_2 , which may depend on $\lambda'_i s$, $\mu'_i s$, Q , σ and d^* , such that for any initial data $(v_0, d_0) \in V \times \mathbf{H}_p^2$ satisfying*

$$\|v_0\|_{\mathbf{H}^1} \leq 1, \quad \|d_0 - d^*\|_{\mathbf{H}^2} \leq 1$$

and

$$\|v_0\| \leq \sigma_1, \quad \|d_0 - d^*\|_{\mathbf{H}^1} \leq \sigma_2,$$

we have

- (i) the problem (1.11)–(1.15) admits a unique global solution (v, d) ,
- (ii) (v, d) enjoys the same long-time behavior as in Theorem 4.2. In addition,

$$\lim_{t \rightarrow +\infty} \mathcal{E}(t) = E(d_\infty) = E(d^*). \quad (5.12)$$

Proof. Without loss of generality, we assume that the constant σ in Definition 5.1 satisfies $\sigma \leq 1$. Throughout the proof, C_i , $i = 1, 2, \dots$ denote generic constants depending only on $\mu'_i s$, $\lambda'_i s$, σ and d^* . By our assumptions, we easily see that

$$\|v(t)\| + \|d(t)\|_{\mathbf{H}^1} \leq C_1, \quad \forall t \geq 0, \quad (5.13)$$

$$\mathcal{A}(0) = \|\nabla v_0\|^2 + \|\Delta d_0 - f(d_0)\|^2 \leq C_2. \quad (5.14)$$

Recalling the proof of Proposition 5.1, we take $R = C_2$ for our current case. The constant C_* in (5.8) can be determined by C_1 and $\mu'_i s, \lambda'_i s$ (cf. Lemma 5.1). Then we set $t_0 = \frac{3}{4}T_{max}(C_2, C_*)$ and take $T^* = t_0$. Finally, the critical constant ε_0 is given by (5.10). It follows from (5.9) that $\mathcal{A}(t)$ is uniformly bounded on $[0, t_0]$, which implies

$$\|v(t)\|_V + \|d(t)\|_{\mathbf{H}^2} \leq C_3, \quad \forall t \in [0, t_0]. \quad (5.15)$$

Next, we extend the local solution to $[0, +\infty)$ by using the Łojasiewicz–Simon approach. Since the minimizer d^* is a critical point of $E(d)$, we take $\psi = d^*$ in the Łojasiewicz–Simon inequality (cf. Lemma 4.5), then the constants $\beta > 0, \theta \in (0, \frac{1}{2})$ are determined by d^* and (4.46) holds.

The proof consists of several steps.

Step 1. In order to apply Proposition 5.1 with $T^* = t_0$, it suffices to show that

$$\mathcal{E}(t) - \mathcal{E}(0) \geq -\varepsilon_0, \quad \forall t \in [0, t_0]. \quad (5.16)$$

Using (5.13) and the Sobolev embedding theorems, we have

$$|E(d_0) - E(d^*)| \leq C_4 \|d_0 - d^*\|_{\mathbf{H}^1},$$

which implies that

$$\begin{aligned} \mathcal{E}(t) - \mathcal{E}(0) &= \frac{1}{2} \|v(t)\|^2 - \frac{1}{2} \|v_0\|^2 + E(d(t)) - E(d_0) \\ &\geq -\frac{1}{2} \|v_0\|^2 + E(d(t)) - E(d^*) + E(d^*) - E(d_0) \\ &\geq -\frac{1}{2} \|v_0\|^2 - C_4 \|d_0 - d^*\|_{H^1} + E(d(t)) - E(d^*). \end{aligned} \quad (5.17)$$

Take

$$\sigma_1 \leq \min \left\{ \varepsilon_0^{\frac{1}{2}}, 1 \right\}, \quad \sigma_2 \leq \min \left\{ \frac{\varepsilon_0}{2C_4}, 1 \right\}.$$

Then by (5.17), it is easy to check (5.16) will be satisfied provided that

$$E(d(t)) - E(d^*) \geq 0, \quad \forall t \in [0, t_0]. \quad (5.18)$$

By the definition of d^* , it reduces to prove that

$$\|d(t) - d^*\|_{\mathbf{H}^1} \leq \sigma, \quad \forall t \in [0, t_0]. \quad (5.19)$$

Actually, we can prove a slightly stronger conclusion such that

$$\|d(t) - d^*\|_{\mathbf{H}^1} < \omega := \frac{1}{2} \min\{\sigma, \beta\}, \quad \forall t \in [0, t_0]. \quad (5.20)$$

Suppose

$$\sigma_2 \leq \frac{1}{4}\omega.$$

We use a contradiction argument. If (5.20) is not true, then by the continuity of d that $d \in C([0, t_0]; \mathbf{H}^1)$, there exists a minimal time $T_0 \in (0, t_0]$, such that

$$\|d(T_0) - d^*\|_{\mathbf{H}^1} = \omega.$$

Observe that

$$\mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + E(d(t)) \geq E(d^*), \quad \forall t \in [0, T_0].$$

First, we consider the trivial case that for some $T \leq T_0$, $\mathcal{E}(T) = E(d^*)$. Then we deduce from the definition of the local minimizer that for $t \geq T$, \mathcal{E} cannot drop and will remain $E(d^*)$. Thus, we infer from the basic energy law (2.7) that the evolution will be stationary and the conclusion easily follows.

In the following, we just assume $\mathcal{E}(t) > E(d^*)$ for $t \in [0, T_0]$. Applying Lemma 4.5 with $\psi = d^*$, we get

$$\begin{aligned} & -\frac{d}{dt}[\mathcal{E}(t) - E(d^*)]^\theta \\ &= -\theta[\mathcal{E}(t) - E(d^*)]^{\theta-1} \frac{d}{dt}\mathcal{E}(t) \\ &\geq \frac{\theta \left(\frac{\mu_4}{2}\|\nabla v\|^2 - \frac{1}{\lambda_1}\|\Delta d - f\|^2 \right)}{C(\|v\|^{2(1-\theta)} + \|\Delta d - f\|)} \\ &\geq C_5(\|\nabla v\| + \|\Delta d - f\|), \quad \forall t \in (0, T_0). \end{aligned}$$

On the other hand, it follows from (1.13) and (5.15) that

$$\begin{aligned} \|d_t\| &\leq \|v \cdot \nabla d\| + \|\Omega d\| + \left| \frac{\lambda_2}{\lambda_1} \right| \|Ad\| - \frac{1}{\lambda_1} \|\Delta d - f\| \\ &\leq C_6(\|v\|_{\mathbf{L}^6} \|\nabla d\|_{\mathbf{L}^3} + \|\nabla v\| \|d\|_{\mathbf{L}^\infty} + \|\Delta d - f\|) \\ &\leq C_7(\|\nabla v\| + \|\Delta d - f\|), \quad \forall t \in [0, t_0]. \end{aligned} \quad (5.21)$$

Consequently,

$$\begin{aligned} \|d(T_0) - d_0\|_{\mathbf{H}^1} &\leq C_8 \|d(T_0) - d_0\|^{\frac{1}{2}} \|d(T_0) - d_0\|_{\mathbf{H}^2}^{\frac{1}{2}} \\ &\leq C_9 \left(\int_0^{T_0} \|d_t(t)\| dt \right)^{\frac{1}{2}} \leq C_{10} [\mathcal{E}(0) - E(d^*)]^{\frac{\theta}{2}} \\ &\leq C_{10} \left(\frac{1}{2} \|v_0\|^2 + C_4 \|d_0 - d^*\|_{\mathbf{H}^1} \right)^{\frac{\theta}{2}} \end{aligned}$$

$$\leq C_{11}(\|v_0\|^\theta + \|d_0 - d^*\|_{\mathbf{H}^1}^{\frac{\theta}{2}}). \quad (5.22)$$

Finally, choosing (also taking the previous assumptions into account)

$$\sigma_1 = \min \left\{ \varepsilon_0^{\frac{1}{2}}, \left(\frac{\omega}{4C_{11}} \right)^{\frac{1}{\theta}}, 1 \right\}, \quad \sigma_2 = \min \left\{ \frac{\varepsilon_0}{2C_4}, \left(\frac{\omega}{4C_{11}} \right)^{\frac{2}{\theta}}, \frac{\omega}{4}, 1 \right\}, \quad (5.23)$$

we can deduce from (5.22) that

$$\begin{aligned} \|d(T_0) - d^*\|_{\mathbf{H}^1} &\leq \|d(T_0) - d_0\|_{\mathbf{H}^1} + \|d_0 - d^*\|_{\mathbf{H}^1} \\ &\leq \frac{\omega}{4} + \frac{\omega}{4} + \frac{\omega}{4} < \omega, \end{aligned}$$

which leads to a contradiction with the definition of T_0 . Thus, (5.20) is true and so is (5.18), which implies that (5.16) is satisfied.

As in the proof of Proposition 5.1, there exists a $t_* \in [\frac{2t_0}{3}, t_0]$, such that $\mathcal{A}(t_*) \leq R$. Then we conclude that $\mathcal{A}(t)$ is uniformly bounded on $[0, t_* + t_0] \supset [0, \frac{5t_0}{3}]$ (with *the same bound* as on $[0, t_0]$). Here, we note the important fact that the bound of $\mathcal{A}(t)$ only depends on R, C_*, t_0 but not on the length of existence interval.

Step 2. Now we take $T^* = \frac{5}{3}t_0$. By the same argument as in Step 1, we can show that

$$\mathcal{E}(t) - \mathcal{E}(0) \geq -\varepsilon_0, \quad t \in [0, T^*].$$

Again, we obtain that $\mathcal{A}(t)$ is uniformly bounded on $[0, T^* + \frac{2}{3}t_0]$ (with *the same bound* as on $[0, t_0]$). By iteration, one can see that the local solution can be extended by a fixed length $\frac{2}{3}t_0$ at each step and $\mathcal{A}(t)$ is uniformly bounded by a constant only depending on R, C_*, t_0 .

Therefore, we can show that (v, d) is indeed a global solution. Moreover, the following uniform estimate holds

$$\|v(t)\|_{\mathbf{H}^1} + \|d(t)\|_{\mathbf{H}^2} \leq K, \quad \forall t \geq 0, \quad (5.24)$$

where K depends on C_1, R, C_*, t_0 . The conclusion (i) is proved.

Step 3. Based on the uniform estimate (5.24), a similar argument to Theorem 4.2 yields that there exists a d_∞ satisfying (4.43)–(4.44), such that

$$\lim_{t \rightarrow +\infty} (\|v(t)\|_V + \|d(t) - d_\infty\|_{\mathbf{H}^2}) = 0, \quad (5.25)$$

with the convergence rate (4.47) (We remark that in (4.47), the Łojasiewicz exponent θ is determined by the limiting function d_∞ , which is *different* from the one we have used in Step 1).

By repeating the argument in Step 1, we are able to show that

$$\|d(t) - d^*\|_{\mathbf{H}^1} \leq \omega, \quad \forall t \geq 0.$$

Then for t sufficiently large, we have

$$\begin{aligned} \|d_\infty - d^*\|_{\mathbf{H}^1} &\leq \|d_\infty - d(t)\|_{\mathbf{H}^1} + \|d(t) - d^*\|_{\mathbf{H}^1} \\ &\leq \frac{3}{2}\omega < \min\{\beta, \sigma\}. \end{aligned} \quad (5.26)$$

Applying Lemma 4.5 again with $d = d_\infty$ and $\psi = d^*$, we obtain

$$|E(d_\infty) - E(d^*)|^{1-\theta} \leq \|-\Delta d^* + f(d^*)\| = 0, \quad (5.27)$$

which together with (5.25) yields (5.12). The proof is complete. \square \square

Remark 5.5. We note that in the assumptions $\|v_0\|_{\mathbf{H}^1} \leq 1$ and $\|d_0 - d^*\|_{\mathbf{H}^2} \leq 1$, the bound 1 is not essential and it can be replaced by any fixed positive constant M . In this case those constants in the proof of Theorem 5.2 may also depend on M . \square

Corollary 5.1 (Nonlinear stability). Suppose that $n = 3$ and the conditions in **Case I** are satisfied. Let $d^* \in \mathbf{H}_p^2$ be a local minimizer of $E(d)$. Then d^* is Lyapunov stable.

Proof. We observe that in the proof of Theorem 5.2, ω can be an arbitrarily small positive constant satisfying $\omega \leq \frac{1}{2} \min\{\sigma, \beta\}$, by our choice of σ_1, σ_2 , we actually have shown that the local minimizer d^* is Lyapunov stable. \square

Remark 5.6. We can see from (5.26) and (5.27) that the asymptotic limit d_∞ obtained in Theorem 5.2 (ii) is also a local minimizer of $E(d)$ (having the same energy level as d^*). Moreover, if d^* is an isolated local minimizer, then $d_\infty = d^*$ and d^* is asymptotically stable. \square

6 Parodi's Relation and Linear Stability

In Section 5.2 we have shown that Parodi's relation can be viewed as a sufficient condition for the nonlinear stability of the Ericksen–Leslie system (1.11)–(1.13). It is still an open problem whether similar result holds true for the original Ericksen–Leslie system (1.1)–(1.3). Alternatively, in this section we shall make a preliminary study to discuss the connection between Parodi's relation and linear stability of the original Ericksen–Leslie system (1.1)–(1.3).

For the sake of simplicity, we assume that

$$\rho = 1, \quad \rho_1 = 0, \quad F = G = 0$$

and the Oseen–Frank energy density function takes the simple form

$$W = \frac{1}{2} |\nabla d|^2.$$

Besides, we are interested in the bulk properties of the Ericksen–Leslie system and consider the problem in the whole space \mathbb{R}^3 , neglecting the boundary effects. We thus simply set the Lagrangian multiplier $\beta = 0$ since β does not enter into the local field equations and can be determined through the boundary conditions, if any, on the director stress (cf. [8]). The nematics usually adopt a constant orientation in uniform shear flow. The analysis in [26, Section 6] shows that for a material that aligns in shear flow the viscous coefficients must satisfy

$$|\mu_5 - \mu_6| \geq |\mu_2 - \mu_3|.$$

Here, we assume that (1.16) and (2.3) are satisfied, then we have

$$|\mu_5 - \mu_6| \geq \mu_3 - \mu_2 > 0. \tag{6.1}$$

Consider the basic uniformly-oriented equilibrium state in which the material is at rest (zero velocity), the orientation is uniformly parallel to a constant unit vector $n = (n_1, n_2, n_3)^T$, the hydrostatic pressure \bar{p} is a constant and the director tension (Lagrangian multiplier) γ is zero. The equilibrium state is disturbed by perturbations with a small amplitude: velocity field v , director $d + n$, pressure $\bar{p} + \tilde{p}$ and director tension $\bar{\gamma}$. After a direct computation, the linearized equations of the Ericksen–Leslie system (1.1)–(1.3) for (v, d) are (cf. e.g., [7])

$$\begin{aligned} \frac{\partial v_i}{\partial t} + \bar{p}_{,i} - \mu_1 n_i n_j n_k n_l v_{j,kl} - \frac{\mu_2 + \mu_5}{2} n_j n_k v_{k,ij} - \frac{\mu_3 + \mu_6}{2} n_i n_k v_{k,jj} - \frac{\mu_4}{2} v_{i,jj} \\ - \frac{\mu_5 - \mu_2}{2} n_j n_k v_{i,kj} - \mu_2 n_j \frac{\partial d_{i,j}}{\partial t} - \mu_3 n_i \frac{\partial d_{j,j}}{\partial t} = 0, \end{aligned} \tag{6.2}$$

$$v_{i,i} = 0, \quad (6.3)$$

$$-\lambda_1 \frac{\partial d_i}{\partial t} - \bar{\gamma} n_i - d_{i,jj} + \frac{\lambda_1 - \lambda_2}{2} n_j v_{i,j} - \frac{\lambda_1 + \lambda_2}{2} n_j v_{j,i} = 0, \quad (6.4)$$

$$d_i n_i = 0. \quad (6.5)$$

We study the behavior of infinitesimal, sinusoidal disturbances by a linear stability analysis. For this purpose, we seek plane wave solutions to the linearized system (6.2)–(6.5) of the following form (see e.g., [9])

$$d = \mathbf{a} e^{\sqrt{-1}(m\nu \cdot x - \omega t)}, \quad (6.6)$$

$$v = \mathbf{b} e^{\sqrt{-1}(m\nu \cdot x - \omega t)}, \quad (6.7)$$

$$\bar{\gamma} = C e^{\sqrt{-1}(m\nu \cdot x - \omega t)}, \quad (6.8)$$

$$\bar{p} = D e^{\sqrt{-1}(m\nu \cdot x - \omega t)}. \quad (6.9)$$

where m is the complex wave number, $\nu = (\nu_1, \nu_2, \nu_3)^T$ is a given unit vector specifying the direction of propagation of the wave and ω is the complex frequency number. \mathbf{a} , \mathbf{b} are two constant vectors and C and D are two constants. Due to the constraint on the unit length of the director (6.5) and the incompressibility condition (6.3), the constant vectors \mathbf{a} and \mathbf{b} satisfy

$$n \cdot \mathbf{a} = 0, \quad \nu \cdot \mathbf{b} = 0. \quad (6.10)$$

Let $0 \leq \theta \leq \frac{\pi}{2}$ be a constant angle such that $\sin \theta = \nu \cdot n$. We consider the in-plane mode and deduce from (6.10) that (cf. [9])

$$\mathbf{a} = A(\nu - n \sin \theta), \quad (6.11)$$

$$\mathbf{b} = B(n - \nu \sin \theta), \quad (6.12)$$

where A and B are two constants. Inserting (6.11) and (6.12) into the linearized system (6.2)–(6.5), after direct but tedious computations, we obtain

$$(m^2 + \sqrt{-1}\lambda_1\omega) A - \frac{\sqrt{-1}m}{2} q(\theta) B = 0, \quad (6.13)$$

$$m\omega p(\theta) A + \left(\frac{m^2 g(\theta)}{2} - \sqrt{-1}\omega \right) B = 0, \quad (6.14)$$

$$C\sqrt{-1}m + \sin \theta \left(\sqrt{-1}w + \frac{\mu_2 + \mu_5}{2} m^2 \cos^2 \theta - \frac{m^2 \mu_4}{2} \right) B - \frac{m^2(\mu_5 - \mu_2)}{2} B \sin^3 \theta - \mu_2 m w A \sin \theta = 0, \quad (6.15)$$

$$D + \sin \theta (m^2 + \sqrt{-1}\lambda_1\omega) A + \frac{\lambda_2 - \lambda_1}{2} \sqrt{-1}m B \sin \theta = 0, \quad (6.16)$$

where

$$g(\theta) = 2\mu_1 \cos^2 \theta \sin^2 \theta + (\mu_3 + \mu_6) \cos^2 \theta + \mu_4 + (\mu_5 - \mu_2) \sin^2 \theta, \quad (6.17)$$

$$p(\theta) = \mu_2 \sin^2 \theta - \mu_3 \cos^2 \theta, \quad (6.18)$$

and

$$\begin{aligned} q(\theta) &= (\lambda_1 + \lambda_2) \cos^2 \theta + (\lambda_1 - \lambda_2) \sin^2 \theta \\ &= (\mu_2 - \mu_3 + \mu_5 - \mu_6) \cos^2 \theta + (\mu_2 - \mu_3 - \mu_5 + \mu_6) \sin^2 \theta. \end{aligned} \quad (6.19)$$

Lemma 6.1. *Suppose that (1.16), (6.1) are satisfied and $\mu_2\mu_3 \geq 0$. There exists a unique real solution $\theta_0 \in [0, \frac{\pi}{2}]$ to the equations*

$$\begin{cases} p(\theta) = 0, \\ q(\theta) = 0, \end{cases} \quad (6.20)$$

if and only if Parodi's relation (1.17) holds.

Proof. If (1.17) holds, we have $q(\theta) = 2p(\theta)$. Then it follows from (6.1) and $\mu_2\mu_3 \geq 0$ that the unique solution to (6.20) is given by

$$\theta_0 = \arctan \left(\sqrt{\frac{\mu_3}{\mu_2}} \right) \in [0, \frac{\pi}{2}].$$

Conversely, suppose $\theta_0 \in [0, \frac{\pi}{2}]$ is the solution to (6.1). We discuss three subcases.

Case 1: $\theta_0 = 0$. It follows from (6.18) that $\mu_3 = 0$. Then we infer from (6.19) that $\mu_2 + \mu_5 - \mu_6 = 0$ and as a result, $\mu_2 + \mu_3 = \mu_6 - \mu_5$.

Case 2: $\theta_0 = \frac{\pi}{2}$. In this case we have $\mu_2 = 0$ and the proof is similar to *Case 1*.

Case 3: $0 < \theta_0 < \frac{\pi}{2}$. In this case it is easy to see that $\mu_2 \neq 0$, $\mu_3 \neq 0$. Since $\sin \theta_0 \neq 0$, $\cos \theta_0 \neq 0$, we deduce from (6.18), (6.19) that

$$(\lambda_1 - \lambda_2)\mu_3 = -(\lambda_1 + \lambda_2)\mu_2,$$

which combined with (1.16) yields (1.17). The proof is complete. \square \square

In the remaining part of this section, we always suppose that (1.16) is valid. We make the following assumptions on the Leslie coefficients $\mu_2, \mu_3, \mu_5, \mu_6$:

$$\mu_6 > 0, \mu_2 > 0, \quad (6.21)$$

$$\mu_5 < \min\{\mu_2, \mu_6\}, \quad (6.22)$$

$$\mu_3 = \mu_6 - \mu_5 + \mu_2 - \epsilon, \quad (6.23)$$

where

$$0 < \epsilon < \min \left\{ \mu_6 - \mu_5, 2\mu_2, \frac{2(\mu_6 - \mu_5)(\mu_2 - \mu_5)}{4\mu_6 - 3\mu_5 + 3\mu_2} \right\}. \quad (6.24)$$

Then we have

Lemma 6.2. *Under the assumptions (6.21)–(6.24), the Leslie coefficients satisfy conditions (2.3) and (6.1), but Parodi's relation (1.17) does not hold. Moreover, there exists a unique solution $\theta_0 \in (0, \frac{\pi}{2})$ such that $p(\theta_0) \neq 0$ and $q(\theta_0) = 0$.*

Proof. It easily follows from (1.16), (6.23) and (6.24) that (2.3) is satisfied. Besides, (6.23) and (6.24) also imply that

$$\mu_6 - \mu_5 < \mu_3 - \mu_2 + 2\mu_2 = \mu_2 + \mu_3,$$

so Parodi's relation (1.17) is not valid in this case. (6.1) can be deduced from (6.22), (6.24) and (2.3) in the sense that

$$|\mu_5 - \mu_6| = \mu_6 - \mu_5 > \mu_3 - \mu_2 > 0.$$

Finally, (2.3) and (6.21) yield that $\mu_2\mu_3 > 0$. Therefore, we can deduce from Lemma 6.1 that there exists an angle

$$\begin{aligned} \theta_0 &= \arctan \sqrt{\frac{\mu_6 - \mu_5 + \mu_3 - \mu_2}{\mu_6 - \mu_5 - \mu_3 + \mu_2}} \\ &= \arctan \sqrt{\frac{2(\mu_6 - \mu_5) - \epsilon}{\epsilon}} \in \left(0, \frac{\pi}{2}\right) \end{aligned} \quad (6.25)$$

such that

$$p(\theta_0) \neq 0 \quad \text{and} \quad q(\theta_0) = 0. \quad (6.26)$$

The proof is complete. \square

We further assume that

$$0 \leq \mu_1 < \frac{1}{4}(2\mu_6 - \mu_5 + \mu_2), \quad (6.27)$$

$$0 \leq \mu_4 < \frac{1}{2}(2\mu_6 - \mu_5 + \mu_2) \cos^2 \theta_0, \quad (6.28)$$

where θ_0 is defined by (6.25). Then we can state the main result of this section:

Theorem 6.1. *Suppose that the Leslie coefficients μ_1, \dots, μ_6 satisfy the assumptions (6.21)–(6.24), (6.27) and (6.28). Then the linearized Ericksen–Leslie system (6.2)–(6.5) admits unstable plane wave solutions.*

Proof. Let θ_0 be the angle obtained in Lemma 6.2 (cf. (6.25)). Taking $\theta = \theta_0$ in the equations (6.13) and (6.14) and using (6.26), we obtain that

$$(m^2 + \sqrt{-1}\lambda_1\omega) A = 0, \quad (6.29)$$

$$m\omega p(\theta_0)A + \left(\frac{m^2 g(\theta_0)}{2} - \sqrt{-1}\omega\right) B = 0. \quad (6.30)$$

Choosing

$$A = 0, \quad B = 1 \quad \text{and} \quad m \in \mathbb{R}, \quad m \neq 0,$$

we deduce from (6.30) that

$$\omega = -\sqrt{-1}\frac{m^2 g(\theta_0)}{2}. \quad (6.31)$$

The (imaginary) constants C and D are determined by (6.15) and (6.16), respectively:

$$C = \frac{m \sin \theta_0}{2} [g(\theta_0) + \mu_2 - \mu_4 + \mu_5 \cos 2\theta_0] \sqrt{-1},$$

$$D = -\frac{\lambda_2 - \lambda_1}{2} \sqrt{-1} m \sin \theta_0.$$

(6.24) implies that

$$\tan^2 \theta_0 = \frac{2(\mu_6 - \mu_5) - \epsilon}{\epsilon} > \frac{2(2\mu_6 - \mu_5 + \mu_2)}{\mu_2 - \mu_5}. \quad (6.32)$$

Consequently, we deduce from (6.27), (6.28) and (6.32) that

$$\begin{aligned} & g(\theta_0) \\ &= 2\mu_1 \cos^2 \theta_0 \sin^2 \theta_0 + (\mu_3 + \mu_6) \cos^2 \theta_0 + \mu_4 \\ & \quad + (\mu_5 - \mu_2) \sin^2 \theta_0 \\ &= 2\mu_1 \cos^2 \theta_0 \sin^2 \theta_0 + \mu_4 + (2\mu_6 - \mu_5 + \mu_2 - \epsilon) \cos^2 \theta_0 \\ & \quad + (\mu_5 - \mu_2) \sin^2 \theta_0 \\ &\leq \cos^2 \theta_0 (2\mu_1 + \mu_4 \sec^2 \theta_0 + [(2\mu_6 - \mu_5 + \mu_2) + (\mu_5 - \mu_2) \tan^2 \theta_0]) \\ &\leq -\frac{(2\mu_6 - \mu_5 + \mu_2)}{2} \cos \theta_0 \\ &< 0. \end{aligned} \quad (6.33)$$

Thus, we obtain the following plane wave solutions $(d, v, \bar{\gamma}, \bar{p})$

$$\begin{aligned} d &= 0, \\ v &= (n - \nu \sin \theta_0) e^{\sqrt{-1} m \nu \cdot x - \frac{m^2 g(\theta_0)}{2} t}, \\ \bar{\gamma} &= C e^{\sqrt{-1} m \nu \cdot x - \frac{m^2 g(\theta_0)}{2} t}, \\ \bar{p} &= D e^{\sqrt{-1} m \nu \cdot x - \frac{m^2 g(\theta_0)}{2} t}, \end{aligned}$$

which are unstable since $g(\theta_0) < 0$. Here, we note that $\nu - n \sin \theta_0 \neq 0$ due to the fact $\theta_0 \in (0, \frac{\pi}{2})$. The proof is complete. \square

Remark 6.1. *Theorem 6.1 indicates that for the nematic liquid crystal flow, if Parodi's relation (1.17) does not hold, the original Ericksen–Leslie system (1.1)–(1.3) system can be (linearly) unstable.* \square

7 Appendices

In this section we provide some detailed computations used in the previous sections.

7.1 Least action principle

The action functional takes the form

$$\mathbb{A}(x) = \int_0^T \int_{\Omega_0} \left[\frac{1}{2} |x_t(X, t)|^2 - \left(\frac{1}{2} |\mathbb{F}^{-T} \nabla_X \mathbb{E} d_0(X)|^2 + \mathcal{F}(\mathbb{E} d_0(X)) \right) \right] J dX dt,$$

where $\Omega_0 = Q$ is the original domain occupied by the material, \mathbb{E} is the deformation tensor satisfying (3.2) and the Jacobian $J = \det \mathbb{F} = 1$. The above expression includes all the kinematic transport property of the molecular director d . With different kinematic transport relations, we will obtain different action functionals, even though the energies may have the same expression in the Eulerian coordinate.

We take any one-parameter family of volume preserving flow map

$$x^\epsilon(X, t) \quad \text{with} \quad x^0 = x, \quad \left. \frac{dx^\epsilon}{d\epsilon} \right|_{\epsilon=0} = y$$

and the volume-preserving constraint $\nabla_x \cdot y = 0$ (or $J^\epsilon = \det \mathbb{F}^\epsilon = 1$). Applying the least action principle, we have

$$\delta_x \mathbb{A} = \left. \frac{d\mathbb{A}(x^\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0$$

such that

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega_0} x_t \cdot y_t dX dt \\ &\quad - \int_0^T \int_{\Omega_0} (\mathbb{F}^{-T} \nabla_X \mathbb{E} d_0) : \left[\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (\nabla_{x^\epsilon} d(x^\epsilon, t)) \right] dX dt \\ &\quad - \int_0^T \int_{\Omega_0} f(\mathbb{E} d_0) \cdot \left(\left. \frac{d\mathbb{E}^\epsilon}{d\epsilon} \right|_{\epsilon=0} d_0 \right) dX dt \\ &:= I_1 + I_2 + I_3, \end{aligned} \tag{7.1}$$

where $\mathbb{E}^\epsilon = \mathbb{E}(x^\epsilon(X, t), t)$. Pushing forward to the Eulerian coordinate, we have

$$\begin{aligned} I_1 &= - \int_0^T \int_{\Omega_0} x_{tt} \cdot y dX dt = - \int_0^T \int_{\Omega_t} \dot{v} \cdot y dx dt \\ &= - \int_0^T \int_{\Omega_t} (v_t + v \cdot \nabla v) \cdot y dx dt, \end{aligned} \quad (7.2)$$

where Ω_t is the domain occupied by the material at time t .

From the definition of \mathbb{E}^ϵ , we have

$$\left. \frac{d\mathbb{E}^\epsilon}{d\epsilon} \right|_{\epsilon=0} d_0 = \left(\frac{1}{2}(\nabla y - \nabla^T y) - \frac{\lambda_2}{2\lambda_1}(\nabla y + \nabla^T y) \right) \mathbb{E} d_0, \quad (7.3)$$

which implies that

$$\begin{aligned} I_2 &= - \int_0^T \int_{\Omega_0} (\mathbb{F}^{-T} \nabla_X \mathbb{E} d_0) : \left(\left. \frac{d(\mathbb{F}^\epsilon)^{-T}}{d\epsilon} \right|_{\epsilon=0} \nabla_X \mathbb{E} d_0 \right) dX dt \\ &\quad - \int_0^T \int_{\Omega_0} (\mathbb{F}^{-T} \nabla_X \mathbb{E} d_0) : \left[\mathbb{F}^{-T} \nabla_X \left(\left. \frac{d\mathbb{E}^\epsilon}{d\epsilon} \right|_{\epsilon=0} d_0 \right) \right] dX dt \\ &= - \int_0^T \int_{\Omega_t} \nabla d : (-\nabla^T y \nabla d) dx dt \\ &\quad - \int_0^T \int_{\Omega_t} \nabla d : \nabla \left[\left(\frac{\nabla y - \nabla^T y}{2} - \frac{\lambda_2}{\lambda_1} \frac{\nabla y + \nabla^T y}{2} \right) d \right] dx dt \\ &= - \int_0^T \int_{\Omega_t} [\nabla \cdot (\nabla d \odot \nabla d)] \cdot y dx dt \\ &\quad + \frac{1}{2} \left(1 - \frac{\lambda_2}{\lambda_1} \right) \int_0^T \int_{\Omega_t} [\nabla \cdot (\Delta d \otimes d)] \cdot y dx dt \\ &\quad - \frac{1}{2} \left(1 + \frac{\lambda_2}{\lambda_1} \right) \int_0^T \int_{\Omega_t} [\nabla \cdot (d \otimes \Delta d)] \cdot y dx dt, \end{aligned} \quad (7.4)$$

and

$$\begin{aligned} I_3 &= - \int_0^T \int_{\Omega_0} f(d) \cdot \left[\left(\frac{1}{2}(\nabla y - \nabla^T y) - \frac{\lambda_2}{2\lambda_1}(\nabla y + \nabla^T y) \right) d \right] dX dt \\ &= \int_0^T \int_{\Omega_t} \left[-\frac{1}{2} \left(1 - \frac{\lambda_2}{\lambda_1} \right) \nabla \cdot (f(d) \otimes d) \right] \cdot y dx dt \\ &\quad + \int_0^T \int_{\Omega_t} \left[\frac{1}{2} \left(1 + \frac{\lambda_2}{\lambda_1} \right) \nabla \cdot (d \otimes f(d)) \right] \cdot y dx dt. \end{aligned} \quad (7.5)$$

Inserting (7.2), (7.4) and (7.5) into (7.1), we arrive at

$$\int_0^T \int_{\Omega_t} [v_t + v \cdot \nabla v + \nabla \cdot (\nabla d \odot \nabla d) - \nabla \cdot \tilde{\sigma}] \cdot y dx dt = 0, \quad (7.6)$$

where

$$\tilde{\sigma} = -\frac{1}{2} \left(1 - \frac{\lambda_2}{\lambda_1} \right) (\Delta d - f(d)) \otimes d + \frac{1}{2} \left(1 + \frac{\lambda_2}{\lambda_1} \right) d \otimes (\Delta d - f(d)). \quad (7.7)$$

Since y is an arbitrary divergence free vector field, we formally derive the momentum equation (Hamiltonian/conservative part) after integration by parts

$$v_t + v \cdot \nabla v = -\nabla P - \nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \tilde{\sigma}, \quad (7.8)$$

where the pressure P serves as a Lagrangian multiplier for the incompressibility of the fluid.

7.2 Maximum dissipation principle

Using the maximum dissipation principle [37–39], we perform a variation on the dissipation functional (half of the total rate of energy dissipation \mathcal{D} (3.7)) with respect to the velocity v in Eulerian coordinates. If $\delta_v(\frac{1}{2}\mathcal{D})$ is set to zero, we will get a weak variational form of the dissipative force balance law equivalent to conservation of momentum. Let $v^\epsilon = v + \epsilon u$, where u is an arbitrary regular function with $\nabla \cdot u = 0$. Then we have

$$\begin{aligned}
0 &= \delta_v\left(\frac{1}{2}\mathcal{D}\right) = \frac{1}{2} \frac{d\mathcal{D}(v^\epsilon)}{d\epsilon} \Big|_{\epsilon=0} \\
&= \frac{\mu_4}{2} \int_Q \nabla v : \nabla u dx + \mu_1 \int_Q d_k A_{kp} d_p d_i \frac{\nabla_i u_j + \nabla_j u_i}{2} d_j dx \\
&\quad - \lambda_1 \int_Q \left(d_t + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d \right) \\
&\quad \cdot \left(u \cdot \nabla d - \frac{\nabla u - \nabla^T u}{2} d + \frac{\lambda_2}{\lambda_1} \frac{\nabla u + \nabla^T u}{2} d \right) dx \\
&\quad + \left(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1} \right) \int_Q A_{ij} d_j \frac{\nabla_i u_k + \nabla_k u_i}{2} d_k dx \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}
I_1 &= -\frac{\mu_4}{2} (\Delta v, u), \\
I_2 &= -\mu_1 \left(\nabla \cdot [d^T A d (d \otimes d)], u \right), \\
I_4 &= -\frac{1}{2} \left(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1} \right) \int_Q \left(u_k \nabla_i (d_k A_{ij} d_j) + u_i \nabla_k (A_{ij} d_j d_k) \right) dx \\
&= -\frac{1}{2} \left(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1} \right) \left[(\nabla \cdot (d \otimes A d), u) + (\nabla \cdot (A d \otimes d), u) \right].
\end{aligned}$$

Using the transport equation (3.6) of d and the incompressibility of u , we infer that

$$\begin{aligned}
I_3 &= \left(\Delta d - f(d), u \cdot \nabla d - \frac{1}{2} \left(1 - \frac{\lambda_2}{\lambda_1} \right) \nabla u d + \frac{1}{2} \left(1 + \frac{\lambda_2}{\lambda_1} \right) \nabla^T u d \right) \\
&= \left(u, -\nabla F(d) + \nabla \cdot (\nabla d \odot \nabla d) - \nabla \frac{|\nabla d|^2}{2} \right) \\
&\quad + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \left(u, \nabla \cdot [(\Delta d - f(d)) \otimes d] \right) \\
&\quad - \left(1 + \frac{\lambda_2}{\lambda_1} \right) \left(u, \nabla \cdot [d \otimes (\Delta d - f(d))] \right) \\
&= (u, \nabla \cdot (\nabla d \odot \nabla d)) - \mu_2 (u, \nabla \cdot (N \otimes d)) - \mu_3 (u, \nabla \cdot (d \otimes N)) \\
&\quad - \eta_5 (u, \nabla \cdot (A d \otimes d)) - \eta_6 (u, \nabla \cdot (d \otimes A d)), \tag{7.9}
\end{aligned}$$

with the coefficients

$$\begin{aligned}
\mu_2 &= \frac{1}{2} (\lambda_1 - \lambda_2), & \mu_3 &= -\frac{1}{2} (\lambda_1 + \lambda_2), \\
\eta_5 &= \frac{1}{2} \left[\lambda_2 - \frac{(\lambda_2)^2}{\lambda_1} \right], & \eta_6 &= -\frac{1}{2} \left[\lambda_2 + \frac{(\lambda_2)^2}{\lambda_1} \right]. \tag{7.10}
\end{aligned}$$

It follows from the above calculations that

$$0 = \frac{1}{2} \frac{d\mathcal{D}}{d\epsilon} \Big|_{\epsilon=0} = (u, \nabla \cdot (\nabla d \odot \nabla d)) - (u, \nabla \cdot \sigma). \tag{7.11}$$

The stress tensor σ is given by

$$\sigma = \mu_1(d^T Ad)d \otimes d + \mu_2 N \otimes d + \mu_3 d \otimes N + \mu_4 A + \tilde{\mu}_5 Ad \otimes d + \tilde{\mu}_6 d \otimes Ad,$$

with constants

$$\begin{aligned} \mu_2 &= \frac{1}{2}(\lambda_1 - \lambda_2), & \mu_3 &= -\frac{1}{2}(\lambda_1 + \lambda_2), \\ \tilde{\mu}_5 &= \frac{1}{2}(\lambda_2 + \mu_5 + \mu_6), & \tilde{\mu}_6 &= \frac{1}{2}(-\lambda_2 + \mu_5 + \mu_6). \end{aligned}$$

Since u is an arbitrary function with $\nabla \cdot u = 0$, we arrive at the dissipative force balance equation

$$0 = -\nabla P - \nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \sigma, \quad (7.12)$$

where the pressure P serves as a Lagrangian multiplier for the incompressibility of the fluid.

7.3 Computation on the time derivative of $\mathcal{A}(t)$

Using (1.11)–(1.13) and integration by parts, due to the periodic boundary conditions, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \mathcal{A}(t) \\ &= -(\Delta v, v_t) + (\Delta d - f, \Delta d_t - f'(d)d_t) \\ &= (\Delta v, v \cdot \nabla v) + (\Delta v, \nabla d \Delta d) - (\nabla \cdot \sigma, \Delta v) + \frac{1}{\lambda_1} \|\nabla(\Delta d - f)\|^2 \\ & \quad - (\Delta d - f, \Delta(v \cdot \nabla d)) + (\Delta d - f, \Delta(\Omega d)) - \frac{\lambda_2}{\lambda_1} (\Delta d - f, \Delta(Ad)) \\ & \quad + \left(\Delta d - f, f'(d) \left[\frac{1}{\lambda_1} (\Delta d - f) + v \cdot \nabla d + \Omega d + \frac{\lambda_2}{\lambda_1} Ad \right] \right). \end{aligned} \quad (7.13)$$

First, we expand the third term on the right-hand side of (7.13):

$$\begin{aligned} & -(\nabla \cdot \sigma, \Delta v) \\ &= - \int_Q \nabla_j \sigma_{ij} \nabla_l \nabla_l v_i dx = - \int_Q \nabla_l \sigma_{ij} \nabla_l \nabla_j v_i dx \\ &= -\mu_1 \int_Q \nabla_l (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_j v_i dx - \mu_4 \int_Q \nabla_l A_{ij} \nabla_l \nabla_j v_i dx \\ & \quad -\mu_2 \int_Q \nabla_l (d_j N_i) \nabla_l \nabla_j v_i dx - \mu_3 \int_Q \nabla_l (d_i N_j) \nabla_l \nabla_j v_i dx \\ & \quad -\mu_5 \int_Q \nabla_l (d_j d_k A_{ki}) \nabla_l \nabla_j v_i dx - \mu_6 \int_Q \nabla_l (d_i d_k A_{kj}) \nabla_l \nabla_j v_i dx. \end{aligned}$$

Using integration by parts and the fact that Ω is antisymmetric, we have

$$\begin{aligned} & -\mu_1 \int_Q \nabla_l (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_j v_i dx \\ &= \mu_1 \int_Q (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_l (A_{ij} + \Omega_{ij}) dx \\ &= \mu_1 \int_Q (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_l A_{ij} dx \\ &= -\mu_1 \int_Q (d_k d_p \nabla_l A_{kp})^2 dx - \mu_1 \int_Q A_{kp} \nabla_l (d_k d_p) d_i d_j \nabla_l A_{ij} dx \end{aligned}$$

$$-\mu_1 \int_Q A_{kp} d_k d_p \nabla_l (d_i d_j) \nabla_l A_{ij} dx. \quad (7.14)$$

By the incompressibility condition $\nabla \cdot v = 0$, we see that

$$\begin{aligned} -\mu_4 \int_Q \nabla_l (A_{ij}) \nabla_l \nabla_j v_i dx &= -\mu_4 \int_Q \nabla_j (A_{ij}) \nabla_l \nabla_l v_i dx \\ &= -\frac{\mu_4}{2} \|\Delta v\|^2. \end{aligned} \quad (7.15)$$

Meanwhile,

$$\begin{aligned} & -\mu_2 \int_Q \nabla_l (d_j N_i) \nabla_l \nabla_j v_i dx - \mu_3 \int_Q \nabla_l (d_i N_j) \nabla_l \nabla_j v_i dx \\ &= \mu_2 \int_Q d_j N_i \Delta (A_{ij} + \Omega_{ij}) dx + \mu_3 \int_Q d_i N_j \Delta (A_{ij} + \Omega_{ij}) dx \\ &= (\mu_2 + \mu_3) \int_Q d_j N_i \Delta A_{ij} dx + (\mu_2 - \mu_3) (N, \Delta \Omega d), \end{aligned} \quad (7.16)$$

and

$$\begin{aligned} & -\mu_5 \int_Q \nabla_l (d_j d_k A_{ki}) \nabla_l \nabla_j v_i dx - \mu_6 \int_Q \nabla_l (d_i d_k A_{kj}) \nabla_l \nabla_j v_i dx \\ &= \mu_5 \int_Q d_j d_k A_{ki} \Delta (A_{ij} + \Omega_{ij}) dx + \mu_6 \int_Q d_j d_k A_{ki} \Delta (A_{ij} - \Omega_{ij}) dx \\ &= (\mu_5 + \mu_6) \int_Q d_j d_k A_{ki} \Delta A_{ij} dx + (\mu_5 - \mu_6) \int_Q d_j d_k A_{ki} \Delta \Omega_{ij} dx \\ &= -(\mu_5 + \mu_6) \int_Q d_j d_k \nabla_l A_{ki} \nabla_l A_{ij} dx - (\mu_5 + \mu_6) \int_Q \nabla_l d_j d_k A_{ki} \nabla_l A_{ij} dx \\ & \quad -(\mu_5 + \mu_6) \int_Q d_j \nabla_l d_k A_{ki} \nabla_l A_{ij} dx + (\mu_5 - \mu_6) (Ad, \Delta \Omega d) \\ &= -(\mu_5 + \mu_6) \int_Q |d_j \nabla_l A_{ji}|^2 dx - (\mu_5 + \mu_6) \int_Q \nabla_l d_j d_k A_{ki} \nabla_l A_{ij} dx \\ & \quad -(\mu_5 + \mu_6) \int_Q d_j \nabla_l d_k A_{ki} \nabla_l A_{ij} dx + (\mu_5 - \mu_6) (Ad, \Delta \Omega d). \end{aligned} \quad (7.17)$$

Next, using the d equation (1.13), we have

$$\begin{aligned} & (\Delta d - f, \Delta(\Omega d)) \\ &= (\Delta d - f, \Delta \Omega d) + 2 \int_Q (\Delta d_i - f_i) \nabla_l \Omega_{ij} \nabla_l d_j dx \\ & \quad + (\Delta d - f, \Omega \Delta d) \\ &= -\lambda_1 \int_Q d_j N_i \Delta \Omega_{ij} dx - \lambda_2 (Ad, \Delta \Omega d) \\ & \quad + 2 \int_Q (\Delta d_i - f_i) \nabla_l \Omega_{ij} \nabla_l d_j dx + (\Delta d - f, \Omega \Delta d) \\ &= -\lambda_1 (N, \Delta \Omega d) - \lambda_2 (Ad, \Delta \Omega d) - \int_Q \nabla_l (\Delta d_i - f_i) \Omega_{ij} \nabla_l d_j dx \\ & \quad + \int_Q (\Delta d_i - f_i) \nabla_l \Omega_{ij} \nabla_l d_j dx, \end{aligned} \quad (7.18)$$

$$\begin{aligned}
& -\frac{\lambda_2}{\lambda_1}(\Delta d - f, \Delta(Ad)) \\
& = \lambda_2(N, \Delta(Ad)) + \frac{(\lambda_2)^2}{\lambda_1}(Ad, \Delta(Ad)) \\
& = \lambda_2 \int_Q N_i \Delta A_{ij} d_j dx + 2\lambda_2 \int_Q N_i \nabla_l A_{ij} \nabla_l d_j dx + \lambda_2(N, A\Delta d) \\
& \quad - \frac{(\lambda_2)^2}{\lambda_1} \int_Q |\nabla_l(A_{ij} d_j)|^2 dx. \tag{7.19}
\end{aligned}$$

Special Cancellations. (i) Due to (1.16), the first term on the right-hand side of (7.18) cancels with the second term of the right-hand side of (7.16) and the second term on the right-hand side of (7.18) cancels with the fourth term of the right-hand side of (7.17). (ii) By Parodi's relation (1.17), the first term of the right-hand side of (7.19) cancels with the first term of the right-hand side of (7.16).

Concerning the fifth term on the right-hand side of (7.13), using the incompressibility of v , the fact $\nabla d \cdot f(d) = \nabla \mathcal{F}(d)$ and integration by parts, we obtain

$$\begin{aligned}
& -(\Delta d - f, \Delta((v \cdot \nabla)d)) \\
& = -(\Delta d - f, \Delta v \cdot \nabla d) - 2 \int_Q (\Delta d_i - f_i) \nabla_l v_j \nabla_l \nabla_j d_i dx - (\Delta d - f, v \cdot \nabla \Delta d) \\
& = -(\Delta v, \nabla d \Delta d) + 2 \int_Q \nabla_j (\Delta d_i - f_i) \nabla_l v_j \nabla_l d_i dx - (\Delta d - f, v \cdot \nabla f).
\end{aligned}$$

Hence,

$$\begin{aligned}
& -(\Delta d - f, \Delta(v \cdot \nabla d)) + (\Delta v, \nabla d \Delta d) \\
& \quad + \left(\Delta d - f, f'(d) \left[\frac{1}{\lambda_1} (\Delta d - f) + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d \right] \right) \\
& = \frac{1}{\lambda_1} \int_Q f'(d) |\Delta d - f|^2 dx - \left(\Delta d - f, f'(d) \left(\Omega d - \frac{\lambda_2}{\lambda_1} A d \right) \right) \\
& \quad + 2 \int_Q \nabla_j (\Delta d_i - f_i) \nabla_l v_j \nabla_l d_i dx - (\Delta d - f, v \cdot \nabla f). \tag{7.20}
\end{aligned}$$

Collecting the above calculations together, we conclude that (4.11) holds.

Acknowledgements

H. Wu was partially supported by NSF of China 11001058, Specialized Research Fund for the Doctoral Program of Higher Education and "Chen Guang" project supported by Shanghai Municipal Education Commission and Shanghai Education Development Foundation. C. Liu and X. Xu were partially supported by NSF grants DMS-0707594 and DMS-1109107. This project began during a long term visit of X. Xu and C. Liu to IMA of University of Minnesota, whose hospitality is gratefully acknowledged. They would like to thank Professors C. Calderer, C. Doering, D. Kinderlehrer, C.-M. Li, F.-H. Lin, E. Titi and C.-Y. Wang for many helpful discussions.

References

- [1] Alouges, F., Ghidaglia, J.-M.: Minimizing Oseen–Frank energy for nematic liquid crystals: algorithms and numerical results. *Ann. Inst. H. Poincaré Phys. Théor.* **66**(4), 411–447 (1997)

- [2] Biot, M.: Variational principles in heat transfer: a unified Lagrangian analysis of dissipative phenomena. Oxford University Press, New York (1970)
- [3] Cavaterra, C., Rocca, E.: On a 3D isothermal model for nematic liquid crystals accounting for stretching terms. *Z. Angew. Math. Phys.* Online first, (2012), DOI: 10.1007/s00033-012-0219-7.
- [4] Cavaterra, C., Rocca, E., Wu, H.: Global weak solution and blow-up criterion of the general Ericksen–Leslie system for nematic liquid crystal flows. Preprint, (2012)
- [5] Chen, Y.-M., Struwe, M.: Existence and partial regularity for heat flow for harmonic maps. *Math. Z* **201**, 83–103 (1989)
- [6] Courant, R., Hilbert, D.: Methods of mathematical physics. Interscience, Vol. **1**, New York (1953)
- [7] Currie, P.: The orientation of liquid crystal by temperature gradients. *Rheol. Acta* **12**, 165–169 (1973)
- [8] Currie, P.: Propagating plane disinclination surfaces in nematic liquid crystals. *Molecular Crystals and Liquid Crystals* **19**, 249–258 (1973)
- [9] Currie, P.: Parodi’s relation as a stability condition for nematics. *Mol. Cryst. Liq. Cryst.* **28**, 335–338 (1974)
- [10] Currie, P.: Decay of weak waves in liquid crystals. *J. Acoust. Soc. Am.* **56**(3), 765–767 (1974)
- [11] de Gennes, P.-G., Prost, J.: The physics of liquid crystals. Oxford Science Publications, Oxford (1993)
- [12] Ericksen, J.: Conservation laws for liquid crystals. *Trans. Soc. Rheol.* **5**, 22–34 (1961)
- [13] Ericksen, J.: Hydrostatic Theory of Liquid Crystal. *Arch. Rational Mech. Anal.* **9**, 371–378 (1962)
- [14] Ericksen, J.: Continuum theory of nematic liquid crystals. *Res. Mechanica* **21**, 381–392 (1987)
- [15] Ericksen, J.: Liquid crystals with variable degree of orientation. *Arch Rational Mech. Anal.* **113**, 97–120 (1991)
- [16] Grasselli, M., Wu, H.: Finite-dimensional global attractor for a system modeling the 2D nematic liquid crystal flow. *Z. Angew. Math. Phys.* **62**, 979–992 (2011)
- [17] Gurtin, M.: An introduction to continuum mechanics. Academic Press, New York (1981)
- [18] Hardt, R., Kinderlehrer, D.: Mathematical questions of liquid crystal theory. The IMA Volumes in Mathematics and its Applications, Vol. **5**, Springer, New York (1987)
- [19] Huang, S.-Z.: Gradient inequalities, with applications to asymptotic behavior and stability of gradient-like systems. *Mathematical Surveys and Monographs*, Vol. **126**, AMS, Providence (2006)
- [20] Hyon, Y., Kwak D.-Y., Liu, C.: Energetic variational approach in complex fluids: maximum dissipation principle. *Discrete Contin. Dyn. Syst.* **26**(4), 1291–1304 (2010)
- [21] Jeffery, G.: The motion of ellipsoidal particles immersed in a viscous fluid. *Roy. Soc. Proc.* **102**, 102–161 (1922)
- [22] Ladyzhenskaya, O.-A., Solonnikov, N.-A., Uraltseva, N.-N.: Linear and quasilinear equations of parabolic type. *Transl. Math. Monographs* Vol. **23**, AMS, Providence (1968)
- [23] Landau, L.-D., Lifshitz, E.-M.: Statistical physics, Course of theoretical physics, Vol. **5**, Butterworths, London (1996)

- [24] Larson, R.-G.: The structure and rheology of complex fluids. Oxford University Press, USA (1998)
- [25] Leslie, F.: Some constitutive equations for anisotropic fluids. *Quart. J. Mech. Appl. Math.* **19**, 357–370 (1966)
- [26] Leslie, F.: Some constitutive equations for liquid crystals. *Arch. Ration. Mech. Anal.* **28**, 265–283 (1968)
- [27] Leslie, F.: Theory of flow phenomena in liquid crystals. in "The Theory of Liquid Crystals", Vol. **4**, 1–81. Academic Press, London-New York (1979)
- [28] Lin, F.-H.: Nonlinear theory of defects in nematic liquid crystal: phase transition and flow phenomena. *Comm. Pure Appl. Math.* **42**, 789–814 (1989)
- [29] Lin, F.-H., Lin, J.-Y., Wang, C.-Y.: Liquid crystal flows in two dimensions. *Arch. Rational Mech. Anal.* **197**, 297–336 (2010)
- [30] Lin, F.-H., Liu, C.: Nonparabolic dissipative system modeling the flow of liquid crystals. *Comm. Pure Appl. Math.* **48**, 501–537 (1995)
- [31] Lin, F.-H., Liu, C.: Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, *Discrete Contin. Dyn. Syst.* **2**, 1–23 (1996)
- [32] Lin, F.-H., Liu, C.: Existence of solutions for the Ericksen–Leslie system, *Arch. Rational Mech. Anal.* **154**(2), 135–156 (2000)
- [33] Lin, F.-H., Wang, C.-Y.: On the uniqueness of heat flow of harmonic maps and hydrodynamic flow of nematic liquid crystals, *Chin. Ann. Math. Ser. B* **31**(6), 921–938 (2010)
- [34] Lin, P., Liu, C., Zhang, H.: An energy law preserving C^0 finite element scheme for simulating the kinematic effects in liquid crystal dynamics. *J. Comput. Phys.* **227**(2), 1411–1427 (2007)
- [35] Liu, C., Shen J., Yang, X.: Dynamics of defect motion in nematic liquid crystal flow: modeling and numerical simulation. *Comm. Comput. Phys.*, **2**, 1184–1198 (2007)
- [36] Mazur, P.: Onsager’s reciprocal relations and the thermodynamics of irreversible processes. *Per. Pol. Chem. Eng.* **41**/2, 197–204 (1997)
- [37] Onsager, L.: Reciprocal relations in irreversible processes I. *Physical Review* **37**, 405–426 (1931)
- [38] Onsager, L.: Reciprocal relations in irreversible processes II. *Physical Review* **38**, 2265–2279 (1931)
- [39] Onsager, L., Machlup, S.: Fluctuations and irreversible processes. *Physical Review* **91**, 1505–1512 (1953)
- [40] Parodi, O.: Stress tensor for a nematic liquid crystal. *Journal de Physique* **31**, 581–584 (1970)
- [41] Rayleigh, L. (previously Strutt, J.-W.): Some general theorems relating to vibrations. *Proc. London Math. Soc.* **IV**, 357–368 (1873)
- [42] Serrin, J.: On the interior of weak solutions of Navier–Stokes equations. *Arch. Ration. Mech. Anal.* **9**, 187–195 (1962)
- [43] Simon, L.: Asymptotics for a class of nonlinear evolution equation with applications to geometric problems. *Ann. of Math.* **118**, 525–571 (1983)
- [44] Sun, H., Liu, C.: On energetic variational approaches in modelling the nematic liquid crystal flows. *Discrete Contin. Dyn. Syst.*, **23**(1&2), 455–475 (2009)

- [45] Temam, R.: Navier–Stokes equations and nonlinear functional analysis, second edition, SIAM, (1995)
- [46] Truesdell, C.: Rational thermodynamics. McGraw-Hill, New York (1969)
- [47] Wu, H.: Long-time behavior for nonlinear hydrodynamic system modelling the nematic liquid crystal flows, *Discrete Contin. Dyn. Syst.* **26**(1), 379–396 (2010)
- [48] Wu, H., Xu, X., Liu, C.: Asymptotic behavior for a nematic liquid crystal model with different kinematic transport properties. *Calc. Var. Partial Differential Equations*. Online first, (2011), DOI: 10.1007/s00526-011-0460-5
- [49] Xu, X., Zhang, Z.-F.: Global regularity and uniqueness of weak solution for the 2D liquid crystal flows. *J. Differential Equations* **252**, 1169–1181 (2012)
- [50] Zhang, J., Gong, X., Liu, C., Wen, W., Sheng, P.: Electrorheological fluid dynamics. *Phys. Rev. Lett.* **101**, 1945032008 (2008)